Department of Meteorology

Sensitivity of VarBC to the Misspecification of Background Error Covariances

Devon Francis* | Alison Fowler | Amos Lawless

Introduction

To correct for observational biases, VarBC extends variational data assimilation to find the analysis of the bias correction coefficients, $\boldsymbol{\beta}$, as well as the state, $\boldsymbol{x}$. The correction is defined via the observation operator, $\boldsymbol{y} = h[\boldsymbol{x}] + c^o[\boldsymbol{x}, \boldsymbol{\beta}]$. The VarBC algorithm depends on the specification of both the background error covariance matrices for the state, $\mathbf{B}_x$ and the bias coefficient, $\mathbf{B}_\beta$. In practice these are never known precisely. We investigate:

- **The sensitivity of the state analysis error covariance $\mathbf{A}_x$ to mis-specifying $\mathbf{B}_\beta$; the sensitivity of the bias correction analysis error covariance $\mathbf{A}_\beta$ to mis-specifying $\mathbf{B}_x$;**
- **Is it better to over- or underestimate $\mathbf{B}_x/\mathbf{B}_\beta$, in order to avoid an analysis that has error covariance greater than the background?**

Theory

In the optimal case, **the analysis error covariance, $\mathbf{A}_v$, for $\boldsymbol{v} = (\boldsymbol{x}, \boldsymbol{\beta})^T$** is given by,

$$\mathbf{A}_{v,\text{opt}}[\mathbf{B}_v] = (\mathbf{I} - \mathbf{K}_v\mathbf{H}_v)\mathbf{B}_v,$$

where the Kalman gain matrix is given by, $\mathbf{K}_v = \mathbf{B}_v\mathbf{H}_v^{\mathrm{T}}\left(\mathbf{H}_v\mathbf{B}_v\mathbf{H}_v^{\mathrm{T}} + \mathbf{R}\right)^{-1}$; $\mathbf{B}_v$ is the true background error covariance matrix for the control vector; $\mathbf{H}_v$ is the linearised observation operator for the control vector; and $\mathbf{R}$ is the observation error covariance matrix.

When the **background error covariance is mis-specified, the analysis error covariance includes a correction term** (see [1])

$$\mathbf{A}_v[\mathbf{B}_v] = \mathbf{A}_{v,\text{opt}}[\mathbf{B}_v^a] + (\mathbf{I} - \mathbf{K}_v^a\mathbf{H}_v)(\mathbf{B}_v - \mathbf{B}_v^a)(\mathbf{I} - \mathbf{K}_v^a\mathbf{H}_v)^{\mathrm{T}},$$

where $\mathbf{B}_v^a$ is the assumed background error covariance matrix for the control vector and $\mathbf{K}_v^a$ is the Kalman gain dependent on $\mathbf{B}_v^a$. Using the result of [1] we find that when we use VarBC, we can separate $\mathbf{A}_v$ into its $\boldsymbol{x}$ and $\boldsymbol{\beta}$ components as follows,

$$\mathbf{A}_x = (\mathbf{I}^n - \mathbf{K}_x^a(\mathbf{H} + \mathbf{C}_x))\mathbf{B}_x^a + (\mathbf{I}^n - \mathbf{K}_x^a(\mathbf{H} + \mathbf{C}_x))\delta\mathbf{B}_x(\mathbf{I}^n - \mathbf{K}_x^a(\mathbf{H} + \mathbf{C}_x))^{\mathrm{T}} + \mathbf{K}_x^a\mathbf{C}_\beta\delta\mathbf{B}_\beta(\mathbf{K}_x^a\mathbf{C}_\beta)^{\mathrm{T}},$$

$$\mathbf{A}_\beta = (\mathbf{I}^r - \mathbf{K}_\beta^a\mathbf{C}_\beta)\mathbf{B}_\beta^a + (\mathbf{I}^r - \mathbf{K}_\beta^a\mathbf{C}_\beta)\delta\mathbf{B}_\beta(\mathbf{I}^r - \mathbf{K}_\beta^a\mathbf{C}_\beta)^{\mathrm{T}} + \mathbf{K}_\beta^a(\mathbf{H} + \mathbf{C}_x)\delta\mathbf{B}_x(\mathbf{K}_\beta^a(\mathbf{H} + \mathbf{C}_x))^{\mathrm{T}},$$

$\mathbf{C}_\beta^o$ and $\mathbf{C}_x^o$ are the linearised bias correction in terms of $\boldsymbol{\beta}$ and $\boldsymbol{x}$; $\delta\mathbf{B}_x = \mathbf{B}_x - \mathbf{B}_x^a$; $\delta\mathbf{B}_\beta = \boldsymbol{B}_\beta - \mathbf{B}_\beta^a$. $\mathbf{K}_x^a$ and $\mathbf{K}_\beta^a$ give the weightings between observations to the prior for the state/bias correction coefficients respectively and are given by,

$$\mathbf{K}_x^a = \mathbf{B}_x^a(\mathbf{H} + \mathbf{C}_x^o)^{\mathrm{T}}((\mathbf{H} + \mathbf{C}_x^o)\mathbf{B}_x^a(\mathbf{H} + \mathbf{C}_x^o)^{\mathrm{T}} + \mathbf{C}_\beta^o\mathbf{B}_\beta^a\mathbf{C}_\beta^{o\mathrm{T}} + \mathbf{R})^{-1},$$

$$\mathbf{K}_\beta^a = \mathbf{B}_\beta^a\mathbf{C}_\beta^{o\mathrm{T}}((\mathbf{H} + \mathbf{C}_x^o)\mathbf{B}_x^a(\mathbf{H} + \mathbf{C}_x^o)^{\mathrm{T}} + \mathbf{C}_\beta^o\mathbf{B}_\beta^a\mathbf{C}_\beta^{o\mathrm{T}} + \mathbf{R})^{-1}.$$

From the equations for $\mathbf{A}_x$ and $\mathbf{A}_\beta$ we find,

- When $\mathbf{K}_x^a(\mathbf{H} + \mathbf{C}_x)$ tends away/towards the identity: $\mathbf{A}_x$ is more/less sensitive to the accuracy of $\mathbf{B}_x$ than the accuracy of $\mathbf{B}_\beta$;
- When $\mathbf{K}_\beta^a\ \mathbf{C}_\beta$ tends away/towards the identity: $\mathbf{A}_\beta$ is more/less sensitive to the accuracy of $\mathbf{B}_\beta$ than the accuracy of $\mathbf{B}_x$.

The '**danger zones**' are defined when $\mathbf{A}_x > \mathbf{B}_x$ or $\mathbf{A}_\beta > \mathbf{B}_\beta$, due to wrongly mis-specifying the assumed background error covariances [1].

Scalar Illustration

Figures 1 and 2 plot the sensitivity of the scalar state analysis error variance, a_x to mis-specifying b_x (self-sensitivity – figure 1) and mis-specifying b_β (cross-sensitivity – figure 2).

Figure 1 varies b_x^a when b_β^a is under- (1a) and over- (1b) estimated.

- a_x is minimised when k_x^a increases and when $\delta b_x = b_x - b_x^a > 0$, ie. when we underestimate b_x^a.
- When b_β^a is overestimated then 'danger zone' decreases.

Figure 2 varies b_β when b_x^a is under- (2a) and over- (2b) estimated.

- a_x is minimised when k_x^a increases and when $\delta b_\beta = b_\beta - b_\beta^a < 0$, ie. when we overestimate b_β^a.
- When b_x^a is underestimated then 'danger zone' decreases.

To avoid the 'danger zone' and minimise a_x it is best to underestimate b_x^a and to overestimate b_β^a.

Scalar equations for a_x and a_β mirror each other, so the opposite is true for a_β, i.e. to avoid the 'danger zone' and minimise a_β it is better to overestimate b_x^a and underestimate b_β^a.

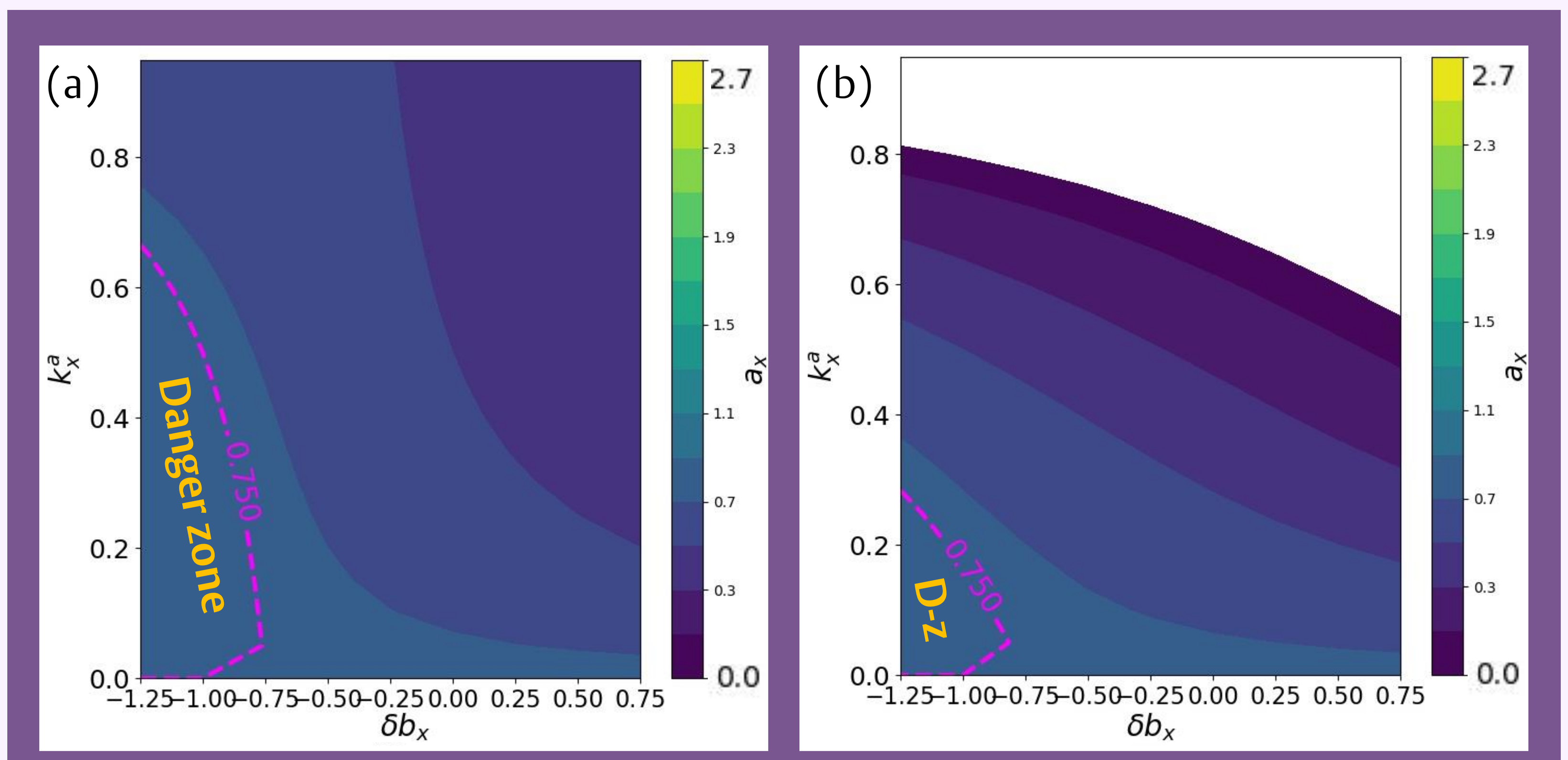

Figure 1: $\boldsymbol{A}_x$ in a scalar illustration, a_x varying both k_x^a and $\delta b = b_x - b_x^a$. $h = c_\beta = b_\beta^a = 1$, $c_x = 0$, $b_x = 0.75$. (a) $b_\beta = 1.5$ and (b) $b_\beta = 0.5$. Pink dashed line shows when $a_x = b_x$.

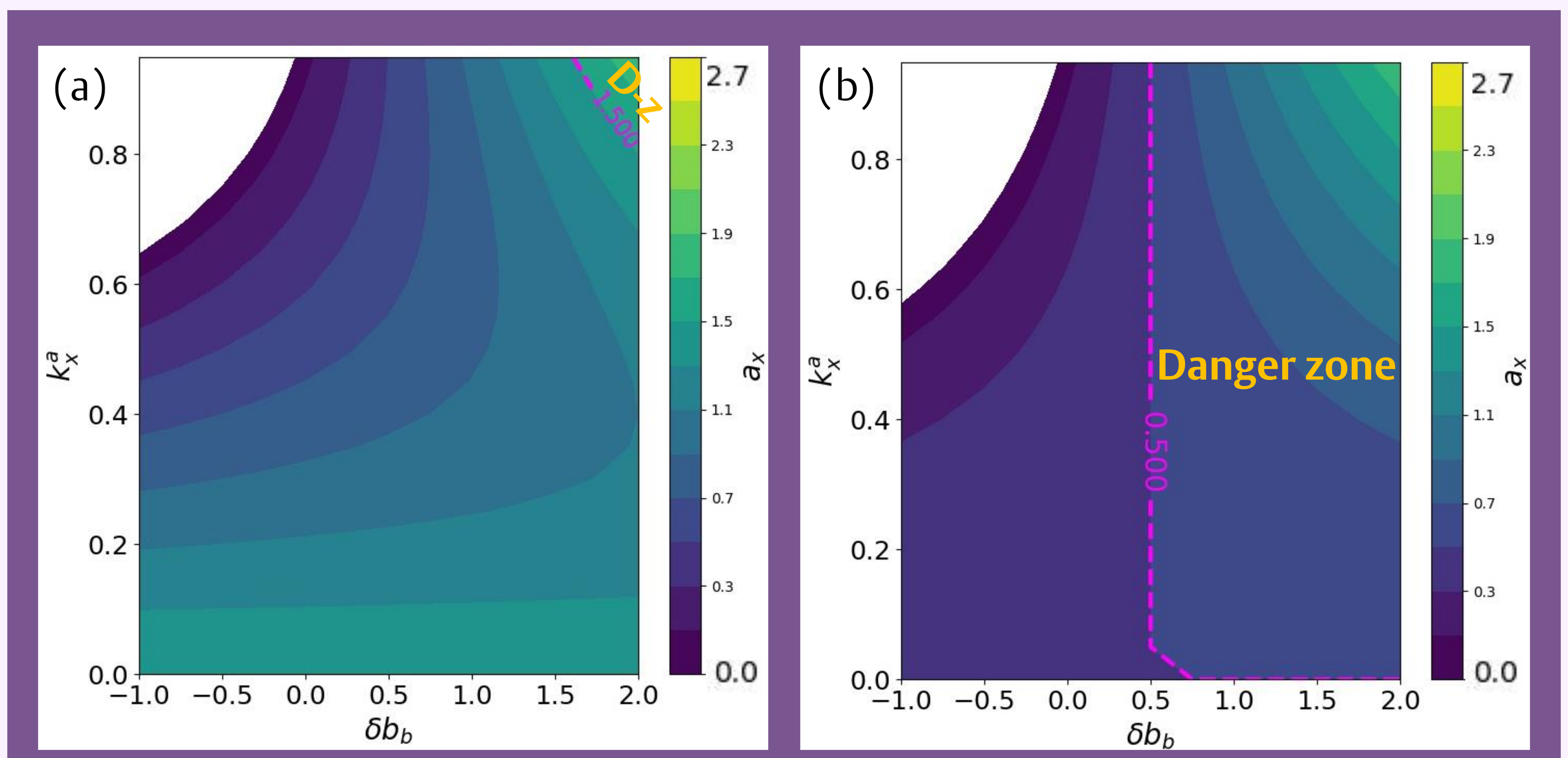

Figure 2: $\boldsymbol{A}_x$ in a scalar illustration, a_x varying both k_x^a and $\delta b_\beta = b_\beta - b_\beta^a$. $h = c_\beta = b_x^a = 1$, $c_x = 0$. (a) $b_x = 1.5$ and (b) $b_x = 0.5$. Pink dashed line shows when $a_x = b_x$

Conclusions

- **Mis-specifying both $\mathbf{B}_x$ and $\mathbf{B}_\beta$ will change $\mathbf{A}_x$ and $\mathbf{A}_\beta$ from their optimal. The sensitivity depends on the gain matrix K.**
- **To avoid $a_x > b_x$, better to underestimate b_x^a and to overestimate b_β^a.**
- **However, to avoid $a_\beta > b_\beta$: best to overestimate b_x^a and to underestimate b_β^a.**
- **These results show how jointly estimating the observation biases, via varBC, changes the criteria for avoiding an analysis of the state and biases, that are less accurate than the background.**

References

1. Eyre, J.R., and Hilton, F.I., *Sensitivity of analysis error covariance to the mis-specification of background error covariance*, Quart. J Roy. Meteor. Soc., 139(671), (2013) pp. 524-533

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*Contact information

- Department of Meteorology, University of Reading, Whiteknights, RG6 6AH
- Email: devon.francis@pgr.reading.ac.uk