A Framework for Exploring Systematic Error in a Quantitatively Complete Measurement Model

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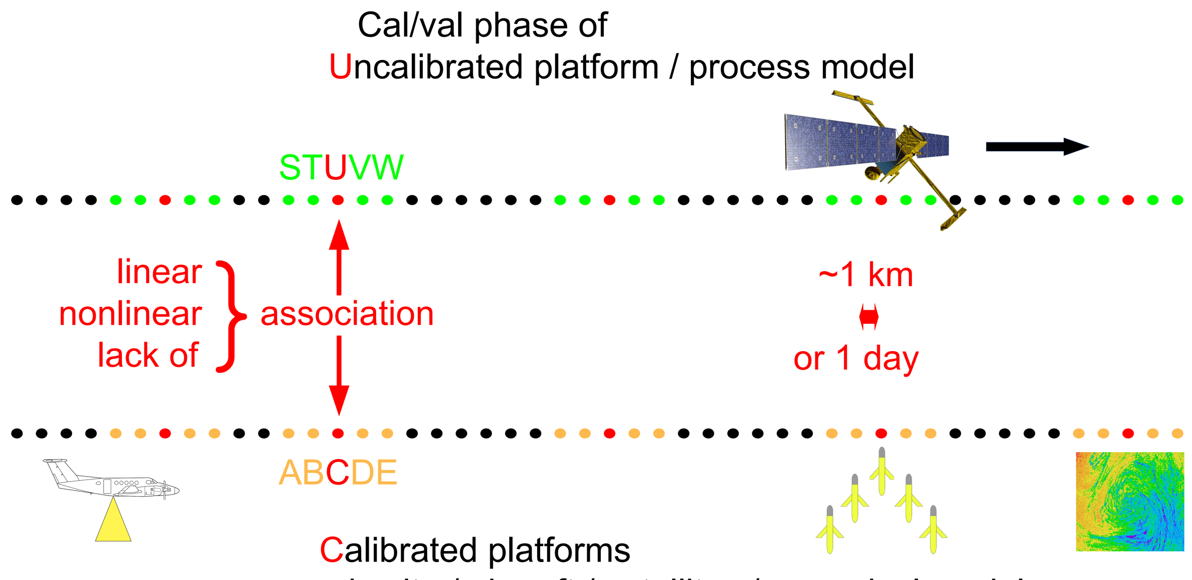
Pearson (1902) seemed convinced of the idea that correlated error could be genuine. Early tropospheric soundings were used by Mahalanobis (1947), who proposed to take a closer look at the genuine nature of correlated error. Yet, in a broad address, Kruskal (1987) seems to suggest that a closer look had not been performed. Today, if the modern definition of dependence imposes no constraint on error correlation (Edelmann et al. 2021), we ask whether the nature of correlated error would benefit from another look, even after so many years:

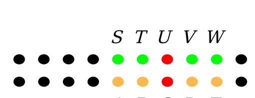
Can systematic and random error be interpreted as genuine signal? Respectively, we submit that "nonlinear association" and "lack of association" are their formal genuine-signal interpretations. First, we develop these ideas out of a purely statistical notion of a "measurement model", or regression model. Then, we examine a playful controlled experiment in the hydrological context.

1. Sampling Signal as a Wave

Our wavelike measurement model employs a collocated sampling strategy that is denoted by the coloured dots (moving left to right). Of interest is the linear, nonlinear, and lack of association between a less familiar "uncalibrated" platform or process model, and one that is more familiar, like an in situ platform. It is notable that wavelike sampling is possible for large datasets that are gridded in space or time; not every scientific discipline has the modelling freedom that comes with such extensive sampling (Donoho 2017).



We employ a trichotomy (e.g., truth + systematic + random error) to describe individual model terms. Model heritage can be traced to an introduction by Pearson (1902) of three observers as a means to identify the errors of each. The geophysical focus on Pearson's approach re-emerged as triple collocation in 1988, and helped motivate our wavelike model. However, instead of three independent observers (or three sets of red dots), method-of-moments solutions are identifiable using "predictive samples" for just two datasets. Collocation of three or more datasets is rare, but a wavelike model still requires successive sampling of two datasets:



$$
\begin{aligned}
S &= \alpha_S + \beta_S\, t + \lambda_S\,(\lambda_T\,(\epsilon + \epsilon_U) + \epsilon_T) + \epsilon_S \\
T &= \alpha_T + \beta_T\, t + \lambda_T\,(\epsilon + \epsilon_U) + \epsilon_T \\
U &= \alpha_U + \beta_U\, t + \epsilon + \epsilon_U \\
V &= \alpha_V + \beta_V\, t + \lambda_V\,(\epsilon + \epsilon_U) + \epsilon_V \\
W &= \alpha_W + \beta_W\, t + \lambda_W\,(\lambda_V\,(\epsilon + \epsilon_U) + \epsilon_V) + \epsilon_W \\
A &= \alpha_A + \beta_A\, t + \lambda_A\,(\lambda_B\,(\epsilon + \epsilon_C) + \epsilon_B) + \epsilon_A \\
B &= \alpha_B + \beta_B\, t + \lambda_B\,(\epsilon + \epsilon_C) + \epsilon_B \\
C &= t + \epsilon + \epsilon_C \\
D &= \alpha_D + \beta_D\, t + \lambda_D\,(\epsilon + \epsilon_C) + \epsilon_D \\
E &= \alpha_E + \beta_E\, t + \lambda_E\,(\lambda_D\,(\epsilon + \epsilon_C) + \epsilon_D) + \epsilon_E
\end{aligned}
$$

S,T,V,W are autocorrelated *Instruments* (proxies) of *U*

A,B,D,E are autocorrelated *Instruments* (proxies) of *C*

or "predictive samples" ("forecasts" or "revcasts")

Here, α, β, and *t* capture linear association between samples, with symmetric first-order autoregressive errors for each dataset separately. Errors are also correlated in ε, which is a term not always included in canonical models (Fuller 2006). Geophysical theory (e.g., equations of motion) allows for a nonlinear relationship between any two samples (like *B-C* or *C-U*). The key question is how a wavelike model accommodates nonlinearity in what looks to be a set of linear relationships?

2. From Error to Signal

To build a model that is complete and nonlinear from the outset, we begin by equating *C*, the calibrated measurements, and *t*, the simplest signal model (ignoring noise for the moment). It seems reasonable to characterize geophysicists as attentive to unresolved scales in *C* when developing their numerical models. Similarly, statisticians are attentive to limitations of the linear model *t*. Thus, when we equate measures and model, a geophysicist might subtract representation error from *C* and a statistician might add equation error to *t*:

Calibrated • $C - \epsilon_C^{REP} \;=\; t + \epsilon_C^{EQ}$

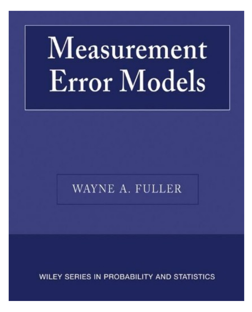
Uncalibrated • $U - \epsilon_U^{REP} \;=\; \alpha_U + \beta_U t + \epsilon_U^{EQ}$

Measure - representation error Model + model error

Ceci n'est pas un éléphant

(en.wikipedia.org/wiki/Asian_elephant)

All models are wrong



Representation error accommodates scales and processes in either *C* or *U* (not in both).

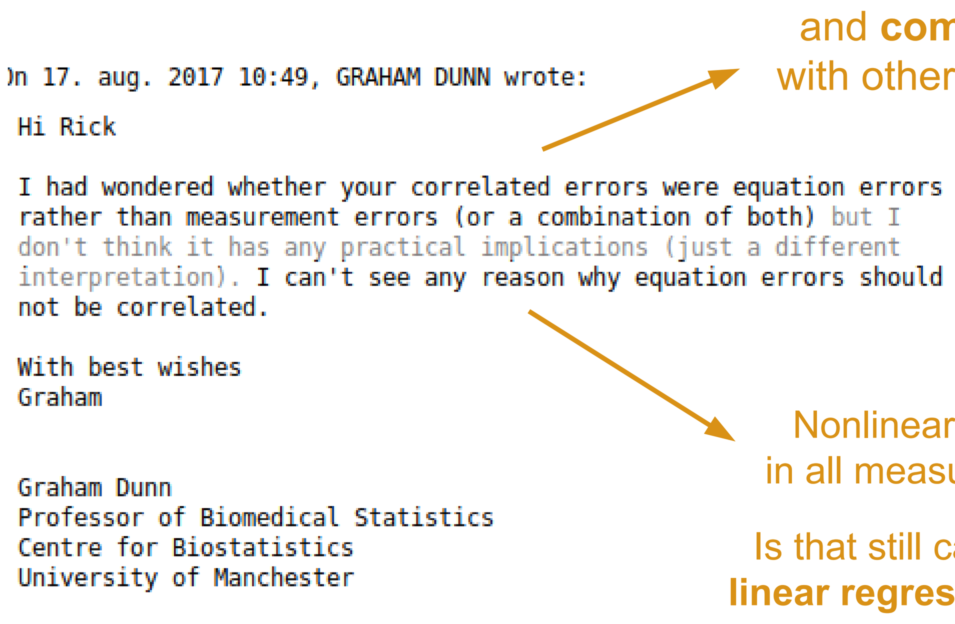
Equation error accommodates nonlinear signal in the relationship between *C* and *U* (via *t*).

Although we write these errors separately, we also need to ask to what extent geophysicists and statisticians are concerned with the same error.

In the literature, it is rare to find a separate equation error or representation error term included in the uncalibrated measurement equation. An accommodation of representation error in both *C* and *U* is sometimes seen as a debate. Perhaps we are just "ripping off the bandaid", in that if we want to accommodate representation error in *C* and *U*, then we might need to accommodate equation error in *C* and *U* as well.

Including a separate equation error in *U* can be justified by its linear relationship to *t* (note that *C* and *U* are equally justified, even though their numerical values differ). Moreover, there seems to be no other way for this model to incorporate the impact of hidden or confounding variables. The only measurements are *C* and *U*, and these may be impacted differently. (There is a third justification that is more practical, as it relates to the model solution.)

Equation error can be **correlated** and **combined** with other errors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On 17. aug. 2017 10:49, GRAHAM DUNN wrote:

 Hi Rick

 I had wondered whether your correlated errors were equation errors
 rather than measurement errors (or a combination of both) but I
 don't think it has any practical implications (just a different
 interpretation). I can't see any reason why equation errors should
 not be correlated.

 With best wishes
 Graham

 Graham Dunn
 Professor of Biomedical Statistics
 Centre for Biostatistics
 University of Manchester
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Nonlinearity in all measures

Is that still called **linear regression?**

In any case, the interpretation of error in terms of signal is challenging, but in 2017, Graham Dunn provided a five-line comment. This took some time to absorb, but basically, he said that equation error could be mixed in with other errors and equation error could be correlated.

If we carry this advice forward, and place measurements on the LHS and model terms on the RHS, then we have linear association, correlated and uncorrelated equation error, and representation error, which is uncorrelated by definition. All are signal components insofar as they are needed to describe *C* and *U* even without measurement error. Without loss of generality, measurement error in *C* and *U* can be divided among the signal terms on the RHS, or written as its own term (as we wish):

$$
\begin{aligned}
C &= t + \epsilon_C^{EQ} + \epsilon_C^{REP} \\
U &= \alpha_U + \beta_U t + \epsilon_U^{EQ} + \epsilon_U^{REP}
\end{aligned}
$$

Measures = linear association
+ correlated equation error
+ uncorrelated equation error
and representation error } Signal
+ measurement error (in all terms) Noise

Since we are interested in terms that can be evaluated numerically, we follow Prof. Dunn's advice and let correlated equation error stand alone and combine uncorrelated equation error and representation error into a separate term:

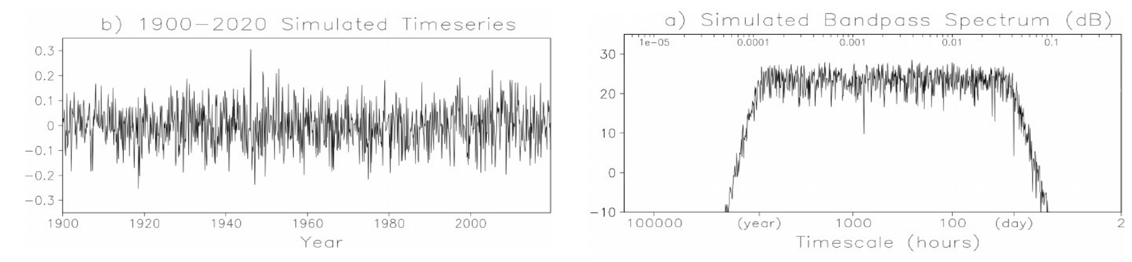
$$
\begin{aligned}
C &= t + \epsilon + \epsilon_C \\
U &= \alpha_U + \beta_U t + \epsilon + \epsilon_U
\end{aligned}
$$

Measures = linear association
+ nonlinear association
+ lack of association

The core of our wavelike measurement model is here. This form makes explicit a "systematic error" term ε that we would call error cross-correlation, but whose *genuine* interpretation is nonlinear association. Similarly, our "random error" terms have the *genuine* interpretation of a lack of association. Although signal defines our interpretation of terms, again, we can conceive of spurious components in each term (https://arxiv.org/abs/2110.08969 is a more formal derivation).

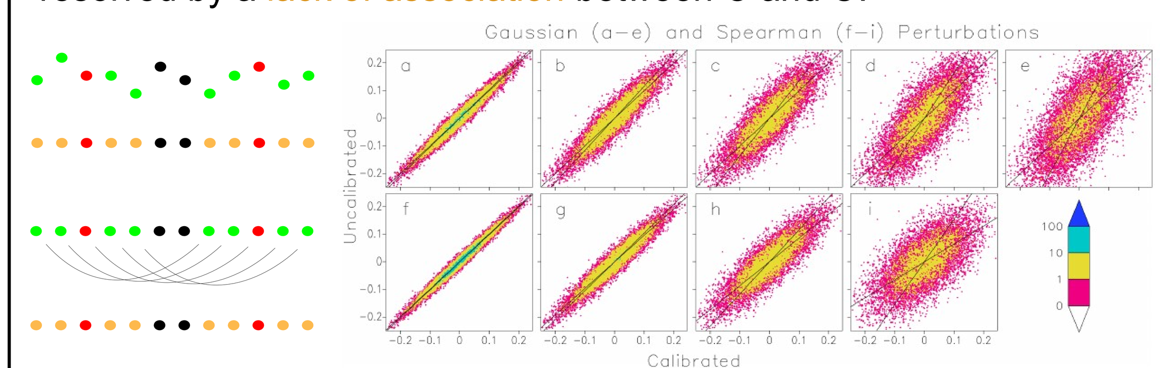
3. A Control Experiment

Returning to Prof. Dunn's question of practical implications, there is no doubt about a different interpretation: measurements, variance, and covariance all have a nonlinear component, and thus, Pearson correlation also has a nonlinear component. Such interpretations can be quantified. By random samples of uniform and Gaussian distributions, we can simulate anomalous river height or river slope at 10000 well separated reaches that have been observed hourly since the turn of the year 1900. The anomaly is such that there is only variability between a day and a year:



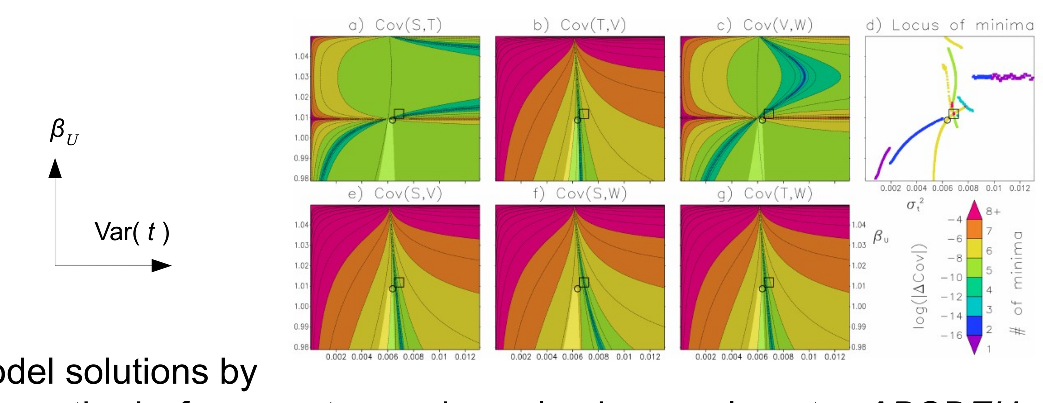
- For each river, signal is slow and wavelike (daily-to-yearly)
- 10000 well-separated rivers sampled (*STUVW* and *ABCDE*) at 1-h intervals
- Perfect calibration : $\beta_U = 1$ (*STUVW* = *ABCDE*)

For each river reach, we generate about a million samples of a uniform random distribution, bandpass filter to get a timeseries in the range -0.2 to 0.2, and from the middle of each timeseries (like the one above), we take 5 consecutive samples to define the slowly varying signal in both *C* and *U*, which are perfectly calibrated. This experiment takes up the challenge of trying to simulate a confounding impact in the presence of measurement error, so two types of perturbations are employed, and both are designed to be resolved by a lack of association between *C* and *U*:

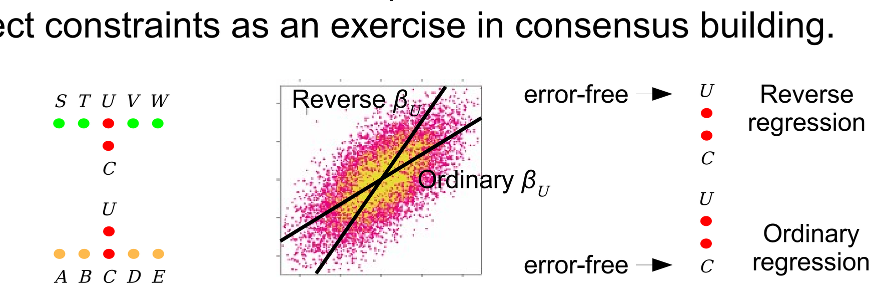


- Gaussian perturbations (only of *U*) are fast and turbulent
- Spearman perturbations (*C* vs *U*) are slow and wavelike (look Gaussian)
- Can we distinguish Gaussian (noise) and Spearman (signal) perturbations?

Gaussian perturbations are added to *U* in varying degrees. Spearman perturbations are a random reordering of signal in *U* that might appear to treat two river reaches as exchangeable, but are meant to simulate a confounding impact without imposing a systematic relationship between *C* and *U*. Although Gaussian and Spearman perturbations might appear to be similar in the upper and lower panels, by design, Spearman perturbations impose no turbulent variations across *STUVW*. Spearman perturbations can be said to perturb *U* alone, but because they are a perturbation of relative order, it follows that if we consider *C* to be unperturbed and equal to the baseline signal, and *U* alone to be perturbed, then **it is only predictive samples involving *STUVW*, but not *ABCDE*, that access both Gaussian and Spearman perturbations, and hence, might distinguish them on this basis.**



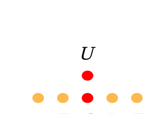
Model solutions by the method of moments require only six sample sets, *ABCDEU* or *CSTUVW*. Each set yields 6 variance equations, 15 covariance equations, and 17 unknowns. Analytic solutions exist for 15 of these unknowns, but the remaining two unknowns are the variance of *t* and β_U, the two key parameters. We minimize the distance of a solution to these two parameters using minima or pathways in each of six covariance equations that involve the variables other than *U* and *C*. Together with the constraint that variance is positive, as expected, we refer to this equitable accommodation of weak or imperfect constraints as an exercise in consensus building.



Wavelike Solutions
- Noisy subsamples
- Consensus solutions
- Better bounds on β_U

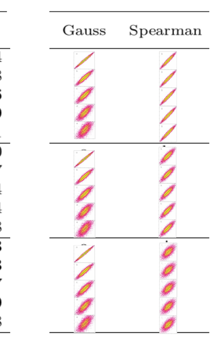
Particle-like Solutions
- No subsamples
- Analytic solutions
- Worse bounds on β_U

Our two wavelike solutions happen to be directly comparable to ordinary and reverse regression solutions that are analytic. So, in the case of our wavelike solutions, *STVW* and *ABDE*, they provide subsamples around *U* and *C* that allow us to estimate Gaussian error, and in the case of analytic solutions by reverse and ordinary regression, we would assume that those samples don't exist and that there is no error in the *U* or *C* variable bracketed by them.



σ_t^2	β_U (O/WAV/R)	Var(C)	LA	NA	UR	Var(U)	LA	NA	UR
6.46	0.95/0.96/1.08	6.57	98.25	0.60	1.15	6.75	87.88	0.58	11.54
6.47	0.95/0.96/1.21	6.57	98.50	0.31	1.20	7.52	79.25	0.27	20.48
6.44	0.95/0.96/1.41	6.57	97.95	0.68	1.37	8.79	67.63	0.51	31.86
6.45	0.95/0.96/1.70	6.57	98.14	0.41	1.45	10.57	56.46	0.25	43.29
6.43	0.95/0.96/2.06	6.57	97.80	0.60	1.60	12.85	46.39	0.31	53.31
6.46	0.85/0.86/1.21	6.57	98.36	0.35	1.28	6.75	70.75	0.34	28.90
6.47	0.85/0.86/1.35	6.57	98.50	0.20	1.30	7.52	63.75	0.18	36.07
6.46	0.85/0.86/1.57	6.57	98.35	0.27	1.38	8.79	54.56	0.20	45.24
6.43	0.85/0.86/1.89	6.57	97.87	0.57	1.56	10.57	45.30	0.36	54.34
6.42	0.85/0.86/2.30	6.57	97.62	0.65	1.73	12.85	37.29	0.33	62.38
6.35	0.63/0.64/1.62	6.57	96.68	1.15	2.17	6.75	39.06	1.12	59.83
6.37	0.63/0.64/1.80	6.57	96.89	0.95	2.16	7.52	35.23	0.84	63.93
6.38	0.63/0.65/2.11	6.57	97.04	0.83	2.13	8.79	30.21	0.62	69.17
6.40	0.63/0.65/2.54	6.57	97.34	0.57	2.09	10.57	25.26	0.36	74.39
6.43	0.63/0.65/3.08	6.57	97.80	0.22	1.97	12.85	20.90	0.11	78.98

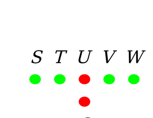
Gauss Spearman



Our lower bound on β_U = 1 (WAV) is **no better than ordinary regression** (O)

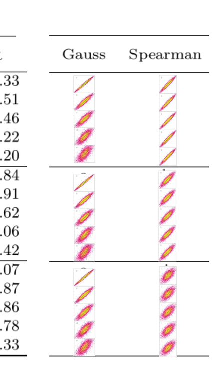
No distinction : mixed Spearman–Gaussian proxy

Linear (LA), nonlinear (NA), and lack of association (UR) results are expressed as normalized *U* and *C* variance budgets, averaged over 100 simulations. By design, *ABDE* are poorly employed, because they don't sample any Gaussian perturbations, which makes this model solution perfectly consistent with ordinary regression, which assumes *C* is error-free.



σ_t^2	β_U (O/WAV/R)	Var(C)	LA	NA	UR	Var(U)	LA	NA	UR
5.79	0.95/1.04/1.08	6.57	88.13	3.28	8.59	6.75	92.47	3.20	4.33
5.30	0.95/1.04/1.21	6.57	80.71	11.22	8.07	7.52	75.68	9.81	14.51
4.88	0.95/1.03/1.41	6.57	74.26	18.41	7.32	8.79	58.78	13.76	27.46
4.42	0.95/1.02/1.70	6.57	67.23	26.30	6.47	10.57	43.44	16.34	40.22
3.94	0.95/1.01/2.06	6.57	60.00	34.18	5.82	12.85	31.36	17.44	51.20
4.68	0.85/1.16/1.21	6.57	71.24	2.57	26.20	6.75	92.67	2.50	4.84
4.30	0.85/1.13/1.35	6.57	65.45	10.69	23.86	7.52	73.75	9.34	16.91
3.88	0.85/1.11/1.57	6.57	59.09	19.23	21.68	8.79	55.00	14.38	30.62
3.55	0.85/1.13/1.89	6.57	54.06	23.80	22.14	10.57	43.15	14.79	42.06
3.05	0.85/1.12/2.30	6.57	46.39	33.32	20.29	12.85	29.56	17.02	53.42
2.65	0.63/1.54/1.62	6.57	40.28	1.63	58.08	6.75	92.34	1.59	6.07
2.41	0.63/1.39/1.80	6.57	36.65	12.03	51.32	7.52	63.62	10.51	25.87
2.27	0.63/1.45/2.11	6.57	34.47	13.13	52.40	8.79	55.33	9.81	34.86
2.03	0.63/1.48/2.54	6.57	30.94	17.78	51.28	10.57	42.18	11.04	46.78
1.57	0.63/1.51/3.08	6.57	23.95	27.50	48.55	12.85	27.62	14.05	58.33

Gauss Spearman



Our upper bound on β_U = 1 (WAV) is **better than reverse regression** (R)

Spearman proxy

Gaussian proxy

By contrast, *STVW* are well employed in sampling Gaussian perturbations. This model solution functions an upper bound on β_U, but provides a better bound than reverse regression. **The lack of association terms in *C* and *U* allow us to distinguish Spearman and Gaussian perturbations**, and we suspect that a model with all three association categories is needed to do so.

4. Conclusions and Links

Recognition of equation error and representation error as partially overlapping signal terms in a wavelike measurement model is proposed (cf. Janssen et al. 2007). A separation of scales that is common in geophysics – slow and wavelike versus fast and turbulent – seems appropriate.

Idealized Spearman and Gaussian perturbations are defined to be slow and fast, respectively. Spearman perturbations are defined as a confounding (random) reordering of one variable relative to the other that appear noisy.

A longitudinal "predictive" sampling, slightly before and after each *C-U* collocation permits a unique identification of model parameters. Based on predictive covariances, a notion of consensus is motivated in solving for a reasonable range of perturbations.

$$
1 = \text{LA} + \text{NA} + \text{UR} = \frac{\sigma_t^2 + \sigma_\epsilon^2 + \sigma_C^2}{Var(C)} = \frac{\beta_U^2\sigma_t^2 + \sigma_\epsilon^2 + \sigma_U^2}{Var(U)}
$$

Annotated references are included in a separate document at https://docs.google.com/document/d/1HRhDJcbdDnaHYth-opOwceL6VK1G_BQ_HdsavvdcE3s

Code is provided to reproduce the "numerical consensus" solutions of this nonlinear model at https://github.com/JuliaAtmosOceanHydro/MeasurementModelDemos