# Towards structure preserving discretizations of stochastic rotating shallow water equations on the sphere

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## Motivation

The motion of geophysical fluids on the globe needs to be modeled to get insights of tomorrow's weather. These forecasts must be precise enough while **remaining computationally affordable**. Ideally they should enable to **estimate likely scenarios** through an ensemble of physically relevant realizations, built from an accurate handling of the model errors that are inescapably introduced due to physical or numerical approximations.

To address these issues, we advocate the use of a **stochastic framework** to represent the action of the many unresolved fast/small-scale processes on the resolved flow component. Simulations are preformed with an adapted **structure preserving numerical model** to maintain numerically the nice properties of the stochastic setting inherited from a transport principle, namely: **mass and energy conservation**. The versatile nature of the stochastic derivation as well as of the proposed numerical scheme makes this framework **suitable for existing dynamical cores** of global numerical weather prediction models.

Numerical results illustrate the energy conservation of the numerical model and the **accuracy of large-scale stochastic simulations** when compared to corresponding deterministic ones. The ability of the random dynamical system to represent model errors is also shown.

## References


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## Stochastic RSW-LU

**Assumption:** fast-slow decomposition of velocity

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{w}(\boldsymbol{X}_t, t)\,\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t, t)\,\mathrm{d}\boldsymbol{B}_t,$$

$\boldsymbol{X}$ Lagrangian displacement, $\boldsymbol{w}$ large-scale velocity (spatially and temporally correlated), $\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t$ highly oscillating unresolved component (also called noise) that is only correlated in space

**Derivation** of consistent stochastic flow models:
i) Holm 15: $\delta \int (\ell(\boldsymbol{w}, q)dt + \langle \boldsymbol{p}, dq + \mathcal{L}_{dx_t} q\rangle_{L^2}) = 0$, $\ell$ Lagrangian, $\mathcal{L}$ Lie derivative of advected quantity $\boldsymbol{q}$, $\boldsymbol{p}$ Lagrange multiplier

ii) Mémin 14 (Location Uncertainty (LU)): stochastic Reynolds transport theorem

**Stochastic rotating shallow water (RSW)-LU:**

$$\mathrm{d}_t \boldsymbol{u} = \Big( -\boldsymbol{u}\cdot\nabla\,\boldsymbol{u} - \boldsymbol{f}\times\boldsymbol{u} - \boldsymbol{g}\nabla\eta \Big)\,\mathrm{d}t + \Big(\frac{1}{2}\,\nabla\cdot\,\nabla\cdot(\boldsymbol{a}\boldsymbol{u})\,\mathrm{d}t - \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\cdot\nabla\,\boldsymbol{u}\Big)$$

$$\mathrm{d}_t h = -\,\nabla\cdot(\boldsymbol{u}h)\,\mathrm{d}t + \Big(\frac{1}{2}\,\nabla\cdot\,\nabla\cdot(\boldsymbol{a}h)\,\mathrm{d}t - \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\cdot\nabla\,h\Big)$$

with variance (matrix) $\boldsymbol{a}$ measuring strength of noise, horizontal velocity component $\boldsymbol{u}$ of 3D large-scale velocity $\boldsymbol{w}$, water depth $\boldsymbol{h}$, surface elevation $\eta = h + B$, bottom topography $\boldsymbol{B}$, Coriolis $\boldsymbol{f}$ and gravity $\boldsymbol{g}$

**Structure-preserving discretization of RSW-LU:**

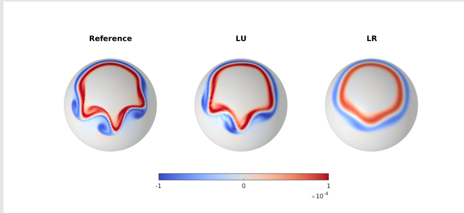
- Use variational RSW scheme for deterministic terms (Bauer & Gay-Balmaz 2019)
- Approximate stochastic terms with standard finite difference operators (Brecht et al. 2021)

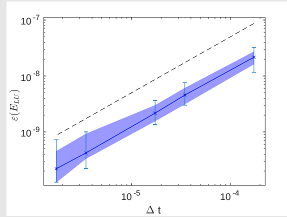
**Properties of LU models:**

- energy brought to system by noise is counterbalanced by corresponding dissipation
- hence: each realization (ensemble member) preserves global energy

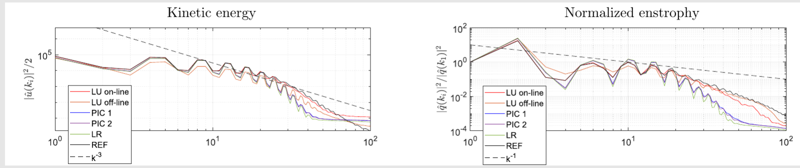
## Numerical results for QG-LU and RSW-LU

- Discretization is energy preserving (in space): noise balanced by dissipation terms
- Spectra of RSW-LU at small scales closer to REF as deterministic PIC



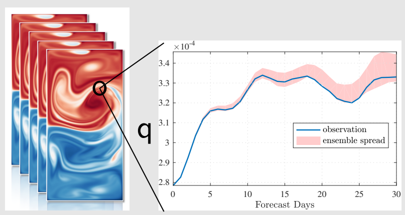

Large scales better represented by stochastic (LU) rather than deterministic (LR) scheme, energy (in space) of each ensemble member preserved ($\approx 1^{st}$ order convergence)



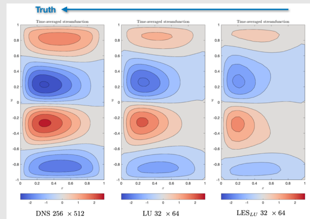

Spectra of kinetic energy and enstrophy after 5 days

## Results for ensemble predictions

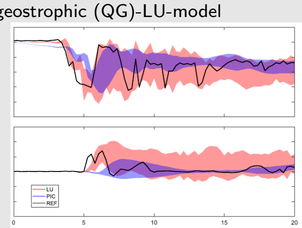
- More reliable ensemble spread with LU than deterministic perturbed initial conditions (PIC)
- LU better captures statistics than PIC
- NOTE: reliable spread important for data assimilation (DA)

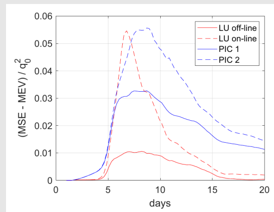



Bauer et al 2020: reliable ensemble forecast of $\boldsymbol{q}$ of quasi-geostrophic (QG)-LU-model




QG-LU captures well stationary moments






Brecht et al. 2021: Left: ensemble spread (at 2 random points) of stochastic LU and deterministic PIC schemes for RSW on sphere. Right Error measure