Hands-on derivation of tangent linear and adjoint codes

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Simple exercise

- Find the adjoint of this nonlinear statement:

$$\mathbf{x} = \alpha\,\mathbf{y} + \beta\,\mathbf{z}^2$$

Note: α and β are constants, **x**, **y**, **z** are variables
Naming conventions

- suffix d for all tangent linear variables
- suffix b for all adjoint variables
- suffix 0 for the trajectory

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Solution (TL)

$$\mathbf{xd} = \alpha\,\mathbf{yd} + 2\beta\,\mathbf{z0}\,\mathbf{zd}$$

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Solution (TL)

$$\mathbf{xd} = \alpha\,\mathbf{yd} + 2\beta\,\mathbf{z0}\,\mathbf{zd}$$

Solution (AD):

$$\mathbf{yb} = 0$$
$$\mathbf{zb} = 0$$
$$\mathbf{yb} = \mathbf{yb} + \alpha\,\mathbf{xb}$$
$$\mathbf{zb} = \mathbf{zb} + 2\beta\,\mathbf{z0}\,\mathbf{xb}$$
$$\mathbf{xb} = 0$$

Simple exercise

Direct

```
real, parameter :: a = ...
real, parameter :: b = ...
subroutine sub(x,y,z)
  real, intent(in)  :: y, z
  real, intent(out) :: x
  !
  x = a*y + b*z**2
```

Tangent linear

```
subroutine sub_tl(x,y,z,z0)
  real, intent(in)  :: y, z
  real, intent(in)  :: z0
  real, intent(out) :: x
  !
  x = a*y + 2*b*z0*z
```

Adjoint

```
subroutine sub_ad(x, y, z, z0)
  real, intent(out)   :: y, z
  real, intent(in)    :: z0
  real, intent(inout) :: x
  !
  y = 0
  z = 0
  y = y + a*x
  z = z + 2*b*z0*x
  x = 0
```

A little more complex exercise

- Find the adjoint of this nonlinear statement using the matrix method:

$$\mathbf{y} = f(\mathbf{a}, \mathbf{b})$$
$$\mathbf{z} = g(\cdots)$$
$$\mathbf{x} = \alpha \mathbf{y} + \beta \mathbf{z}^2$$

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$$\mathbf{z} = g(\cdots)$$
$$\mathbf{x} = \alpha \mathbf{y} + \beta \mathbf{z}^2$$

Solution (TL)

$$\mathbf{z0} = g(\cdots)$$
$$\mathbf{yd} = f\_tl(\mathbf{a}, \mathbf{b}, \mathbf{a0}, \mathbf{b0})$$
$$\mathbf{zd} = g\_tl(\cdots)$$
$$\mathbf{xd} = \alpha \mathbf{yd} + 2\beta\, \mathbf{z0}\, \mathbf{zd}$$

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$$\mathbf{x} = \alpha \mathbf{y} + \beta \mathbf{z}^2$$

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$$\mathbf{yd} = f\_tl(\mathbf{a}, \mathbf{b}, \mathbf{a0}, \mathbf{b0})$$
$$\mathbf{zd} = g\_tl(\cdots)$$
$$\mathbf{xd} = \alpha \mathbf{yd} + 2\beta\, \mathbf{z0}\, \mathbf{zd}$$

Solution (AD):

$$\mathbf{z0} = g(\cdots)$$
$$\mathbf{yb} = 0\,;\, \mathbf{zb} = 0$$
$$\mathbf{yb} = \mathbf{yb} + \alpha\, \mathbf{xb}$$
$$\mathbf{zb} = \mathbf{zb} + 2\beta\, \mathbf{z0}\, \mathbf{xb}$$
$$\mathbf{xb} = 0$$
$$\cdots = g\_ad(\mathbf{zb}, \cdots)$$
$$\mathbf{ab} = 0;\, \mathbf{bb} = 0$$
$$\mathbf{ab} = \mathbf{ab} + f\_ad(\mathbf{yb}, \mathbf{a0}, \mathbf{b0})$$
$$\mathbf{bb} = \mathbf{bb} + f\_ad(\mathbf{yb}, \mathbf{a0}, \mathbf{b0})$$

A definitly more complex exercise

- Find the adjoint of this nonlinear statement:

$$
\begin{array}{l}
\mathbf{y}(:) = f(\cdots) \\
\mathbf{z}(:) = g(\cdots) \\
\mathbf{x}(1) = \gamma \\
\text{for } i = 2, n \\
\qquad \mathbf{x}(i) = \alpha\,\mathbf{x}(i-1) + \beta\,\mathbf{y}(i)\,\mathbf{z}(i) \\
\text{endfor}
\end{array}
$$

A definitly more complex exercise

- Find the adjoint of this nonlinear statement:

$$\mathbf{y}(:) = f(\cdots)$$
$$\mathbf{z}(:) = g(\cdots)$$
$$\mathbf{x}(1) = \gamma + \beta\,\mathbf{y}(1)\,\mathbf{z}(1)$$

for $i = 2, n$

$$\mathbf{x}(i) = \alpha\,\mathbf{x}(i-1) + \beta\,\mathbf{y}(i)\,\mathbf{z}(i)$$

endfor

Solution (TL)

$$\mathbf{y0}(:) = f(\cdots)$$
$$\mathbf{z0}(:) = g(\cdots)$$

$$\mathbf{yd}(:) = f\_tl(\cdots)$$
$$\mathbf{zd}(:) = g\_tl(\cdots)$$
$$\mathbf{xd}(1) = \beta\,\mathbf{yd}(1)\,\mathbf{z0}(1) + \beta\,\mathbf{y0}(1)\,\mathbf{zd}(1)$$

for $i = 2, n$

$$\mathbf{xd}(i) = \alpha\,\mathbf{xd}(i-1) + \beta\,\mathbf{yd}(i)\,\mathbf{z0}(i) + \beta\,\mathbf{y0}(i)\,\mathbf{zd}(i)$$

endfor

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endfor

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for $i = 2, n$

$$\mathbf{xd}(i) = \alpha\,\mathbf{xd}(i-1) + \beta\,\mathbf{yd}(i)\,\mathbf{z0}(i) + \beta\,\mathbf{y0}(i)\,\mathbf{zd}(i)$$

endfor

Solution (AD):

$$\mathbf{y0}(:) = f(\cdots)$$
$$\mathbf{z0}(:) = g(\cdots)$$

$$\mathbf{yb}(:) = \mathbf{zb}(:) = 0$$

for $i = n, 2$

$$\mathbf{zb}(i) = \mathbf{zb}(i) + \beta\ \mathbf{y0}(i)\ \mathbf{xb}(i)$$
$$\mathbf{yb}(i) = \mathbf{yb}(i) + \beta\ \mathbf{z0}(i)\ \mathbf{xb}(i)$$
$$\mathbf{xb}(i-1) = \mathbf{xb}(i-1) + \alpha\,\mathbf{xb}(i)$$
$$\mathbf{xb}(i) = 0$$

endfor

$$\mathbf{zb}(1) = \mathbf{zb}(1) + \beta\ \mathbf{y0}(1)\ \mathbf{xb}(1)$$
$$\mathbf{yb}(1) = \mathbf{yb}(1) + \beta\ \mathbf{z0}(1)\ \mathbf{xb}(1)$$
$$\mathbf{xb}(1) = 0$$
$$\cdots = g\_ad(\mathbf{zb}, \cdots)$$
$$\cdots = f\_ad(\mathbf{yb}, \cdots)$$

Absolute value

- Find the adjoint of this nonlinear statement:

$$\mathbf{x} = abs(\mathbf{y})$$

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Solution 1 (DIR):

$$\begin{aligned} &\textit{if } \mathbf{y} > 0 \textit{ then} \\ &\quad \mathbf{x} = \mathbf{y} \\ &\textit{else} \\ &\quad \mathbf{x} = -\mathbf{y} \end{aligned}$$

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$$\textit{if } \mathbf{y} > 0 \textit{ then}$$
$$\mathbf{x} = \mathbf{y}$$
$$\textit{else}$$
$$\mathbf{x} = -\mathbf{y}$$

Solution 2 (DIR):

$$\text{if } -\gamma < \mathbf{y} < \gamma \text{ then}$$
$$\mathbf{x} = \alpha \mathbf{y}^2 + \varepsilon$$
$$\text{else}$$
$$\mathbf{x} = \mathbf{y}$$

with α, γ and ε adjusted so **x** and its derivative are continuous

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with α, γ and ε adjusted so **x** and its derivative are continuous

Solution 2 (TL):

$$\begin{aligned} &\text{if } -\gamma < \mathbf{y0} < \gamma \text{ then} \\ &\quad \mathbf{xd} = 2\alpha\, \mathbf{y0}\, \mathbf{yd} \\ &\text{else} \\ &\quad \mathbf{xd} = \mathbf{yd} \end{aligned}$$