Introduction to chaos

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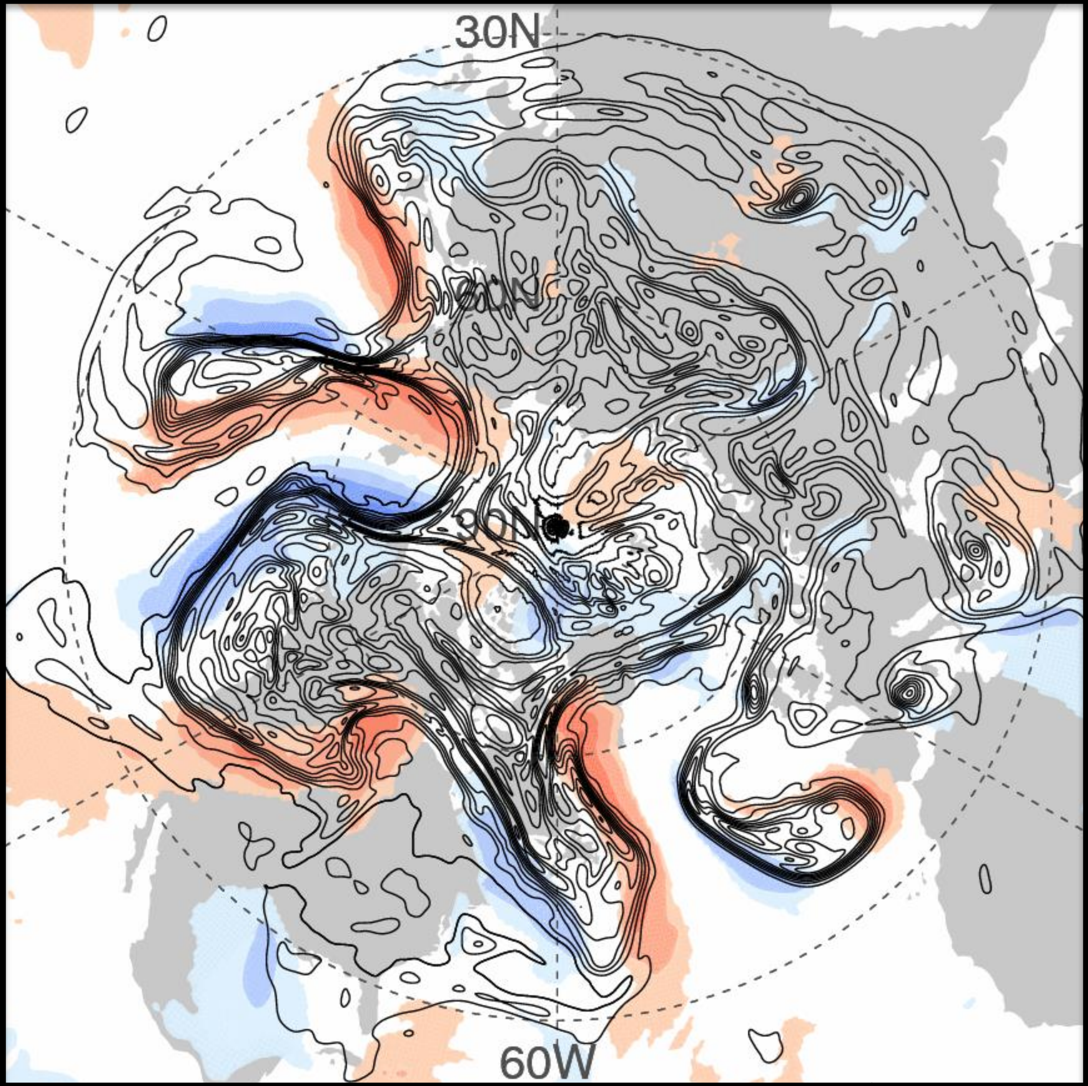

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Vilhelm Bjerknes (1862-1951)

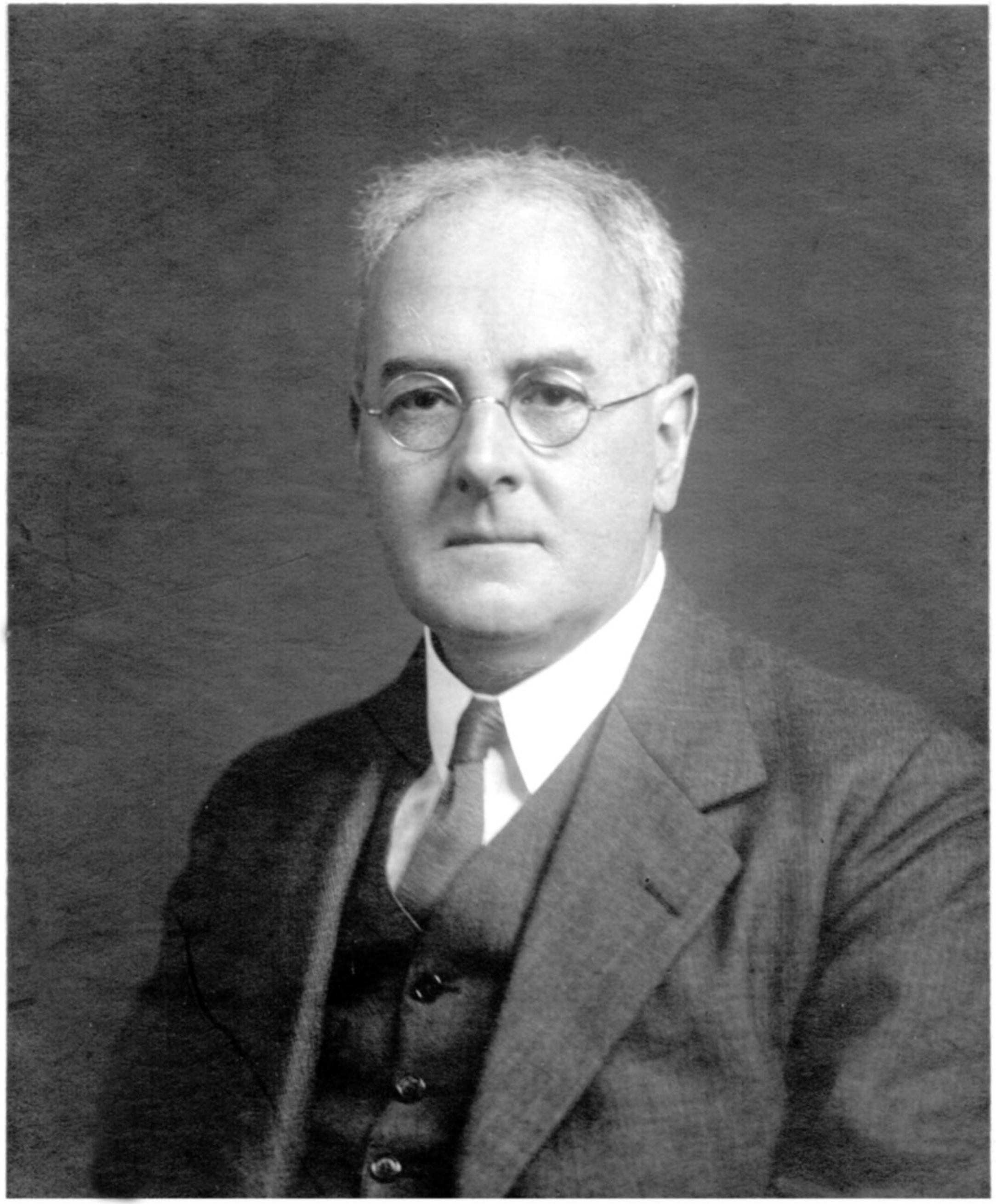
“Founding father of modern weather forecasting”

Norwegian physicist who proposed weather forecasting as a deterministic initial value problem based on the laws of physics

Lewis Fry Richardson (1881-1953)

English scientist who produced the first numerical weather forecast

- Forecast for 20 May 1910 1pm by direct computation of the solutions to simplified flow equations using input data taken at 7am
- Forecast predicted rise in surface pressure by 145 hPa in 6 hours → dramatic failure
- A posteriori: failure to apply smoothing to data to filter out unphysical waves

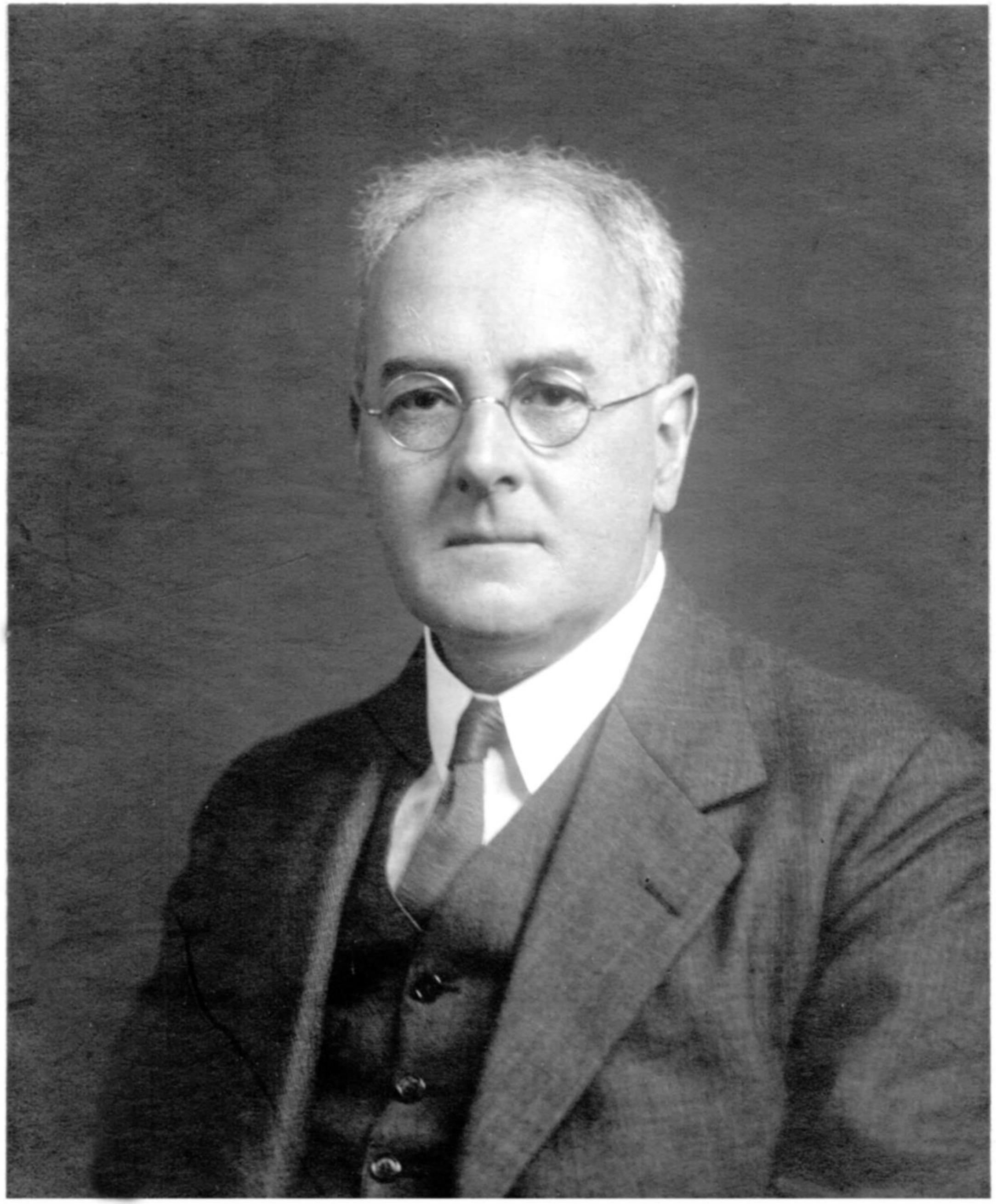


L. F. Richardson, 1931

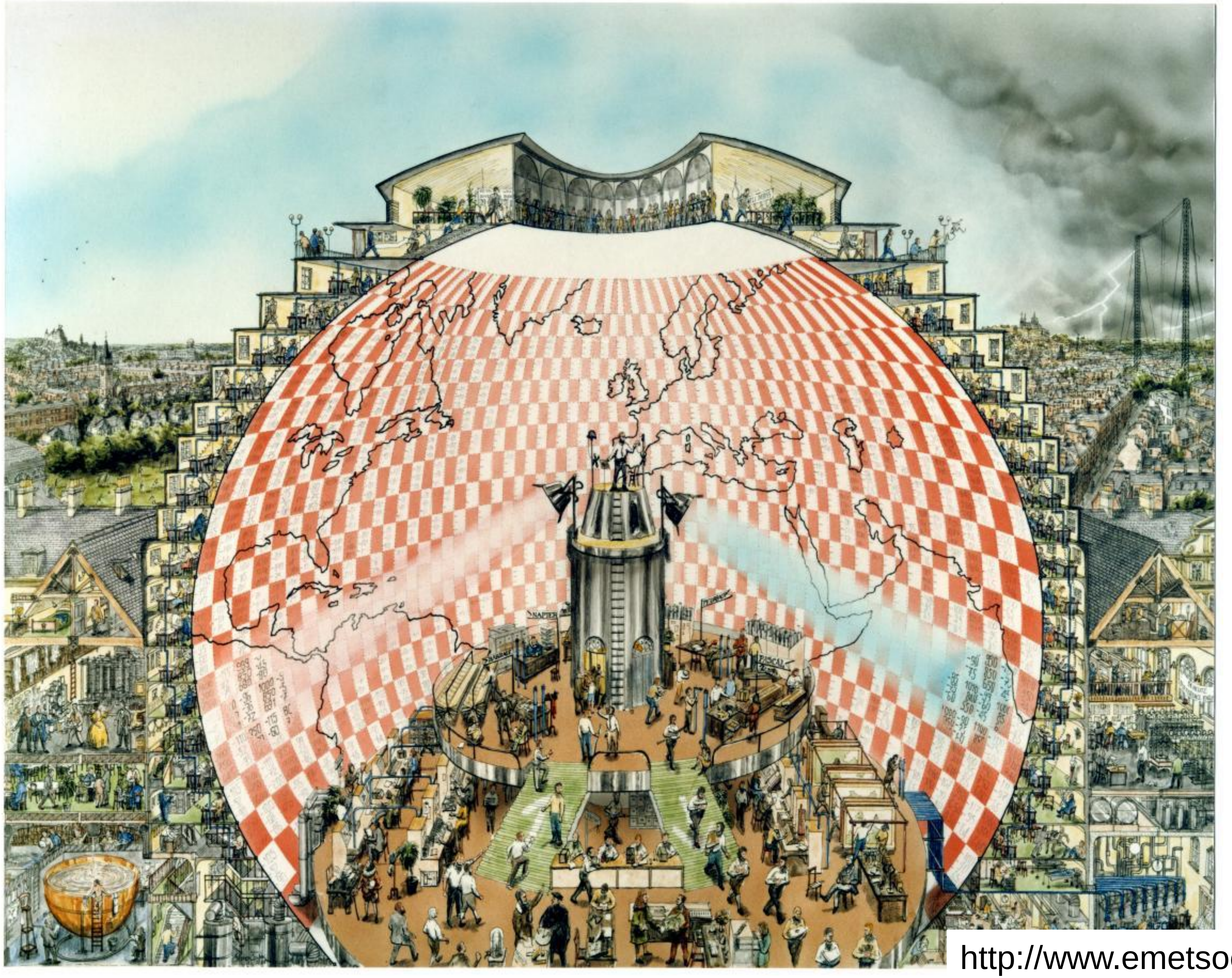
Lewis Fry Richardson (1881-1953)

Author of "Weather Prediction by Numerical Process" (1922)

Richardson devised a method of **solving the mathematical equations** that describe atmospheric flow **by dividing the globe into cells and specifying the dynamical variables at the centre of each cell.** In Chapter 11 of his book, he presents what he calls a 'fantasy', describing in detail **his remarkable vision of an enormous building**, **a fantastic forecast factory**.



L. F. Richardson, 1931



http://www.emetsoc.org/resources/rff

Henry Poincaré (1854-1912)

French mathematician, physicist and philosopher of science

- Fundamental contributions to pure and applied mathematics
- Studying the three-body problem, he became the first person to discover a chaotic deterministic system
- Laid foundations for modern chaos theory

“Why have meteorologists such difficulty in predicting the weather with any certainty? Why is it that showers and even storms seem to come by chance ... a tenth of a degree (C) more or less at any given point, and the cyclone will burst here and not there, and extend its ravages over districts that it would otherwise have spared. **If (the meteorologists) had been aware of this tenth of a degree, they could have known (about the cyclone) beforehand, but the observations were neither sufficiently comprehensive nor sufficiently precise,** and that is the reason why it all seems due to the intervention of chance”

Poincaré, 1909

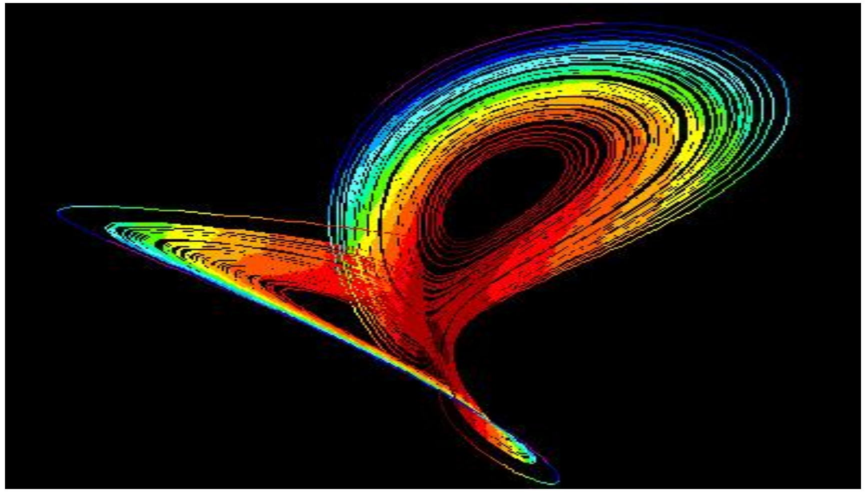
Sensitive dependence on initial conditions

If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of the same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation *approximately*. If that enabled us to predict the succeeding situation with *the same approximation*, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that **small differences in the initial conditions produce very great ones in the final phenomena**. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon.

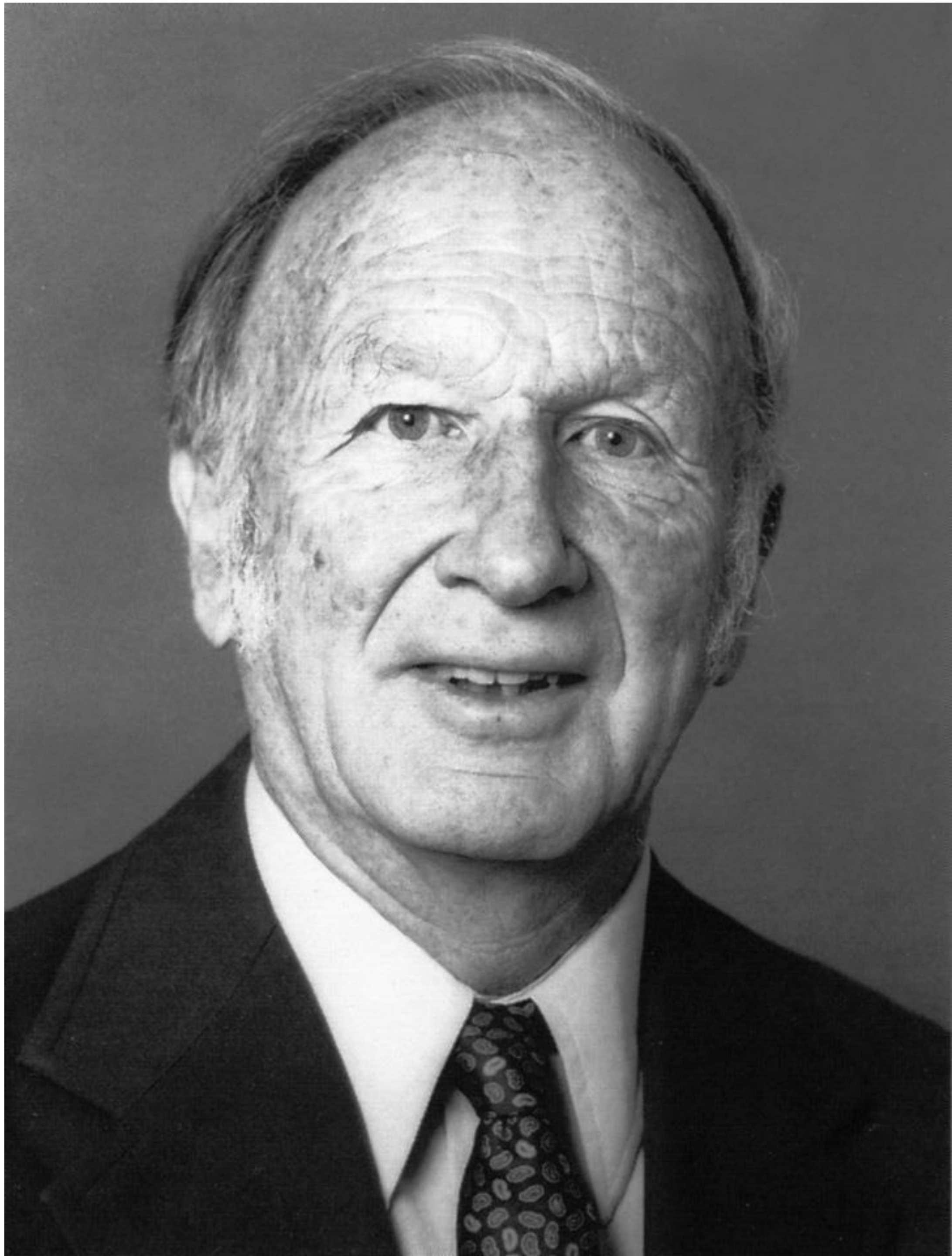
Poincaré, 1903 “Science and Method”

Edward Lorenz (1917 –2008)

"… one flap of a sea-gull's wing may forever change the future course of the weather"



The Lorenz (1963) attractor: a prototype chaotic model



Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

1. Introduction

Certain hydrodynamical systems exhibit steady-state flow patterns, while others oscillate in a regular periodic fashion. Still others vary in an irregular, seemingly haphazard manner, and, even when observed for long periods of time, do not appear to repeat their previous history.

These modes of behavior may all be observed in the familiar rotating-basin experiments, described by Fultz, *et al.* (1959) and Hide (1958). In these experiments, a cylindrical vessel containing water is rotated about its axis, and is heated near its rim and cooled near its center

Thus there are occasions when more than the statistics of irregular flow are of very real concern.

In this study we shall work with systems of deterministic equations which are idealizations of hydrodynamical systems. We shall be interested principally in nonperiodic solutions, i.e., solutions which never repeat their past history exactly, and where all approximate repetitions are of finite duration. Thus we shall be involved with the ultimate behavior of the solutions, as opposed to the transient behavior associated with arbitrary initial conditions.

A closed hydrodynamical system of finite mass may

Lorenz (1963, JAS)

Deterministic chaos

$$\dot{X} = -\sigma X + \sigma Y$$
$$\dot{Y} = -XZ + rX - Y$$
$$\dot{Z} = XY - bZ$$

"… our results .. indicate that prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly. In view of the inevitable inaccuracy and incompleteness of weather observations, precise very-long-range forecasting would seem to be non-existent."

What is deterministic chaos?

A physical system that

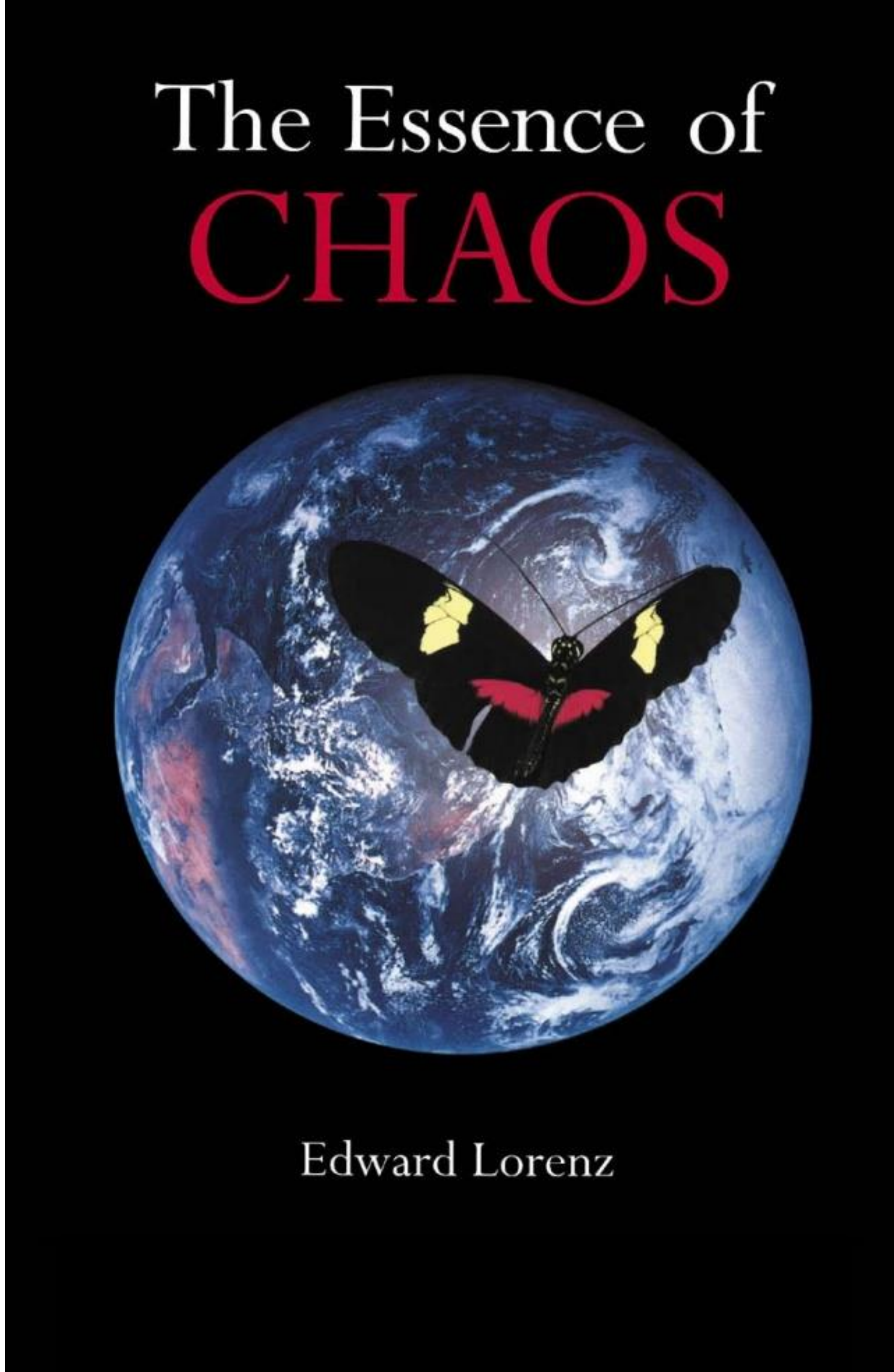
- follows deterministic rules (absence of randomness)
- *but appears* to behave randomly; it *looks* random
- is sensitive dependent on the initial conditions
- Is nonlinear, dissipative and at least 3-dimensional
- growth of perturbations is flow dependent

$$\frac{dX}{dt} = F[X] \quad \text{is a nonlinear system}$$

$$\Rightarrow \frac{d\delta X}{dt} = \frac{dF}{dX} \delta X \equiv J \quad \delta X$$

Since F is a nonlinear function of X

$$\Rightarrow J = J(X)$$



Brief glossary (after E. Lorenz)

Nonlinear system: A system in which alterations in an initial state need not produce proportional alterations in subsequent states

Dissipative system: A dynamical system in which the temporal evolution of any set of points of finite volume in phase space leads to a set of smaller volume

Attractor: In a dissipative system, a limit set that is not contained in any larger limit set, and from which no orbits (trajectories) emanate

Strange attractor: An attractor with a fractal structure (dimension of the set is not a whole number)

Sensitive Dependence: The property characterising an orbit (trajectory) if most other orbits that pass close to it at some point do not remain close to it as time advances

Chaos: The property that characterises a dynamical system in which most orbits (trajectories) exhibit sensitive dependence

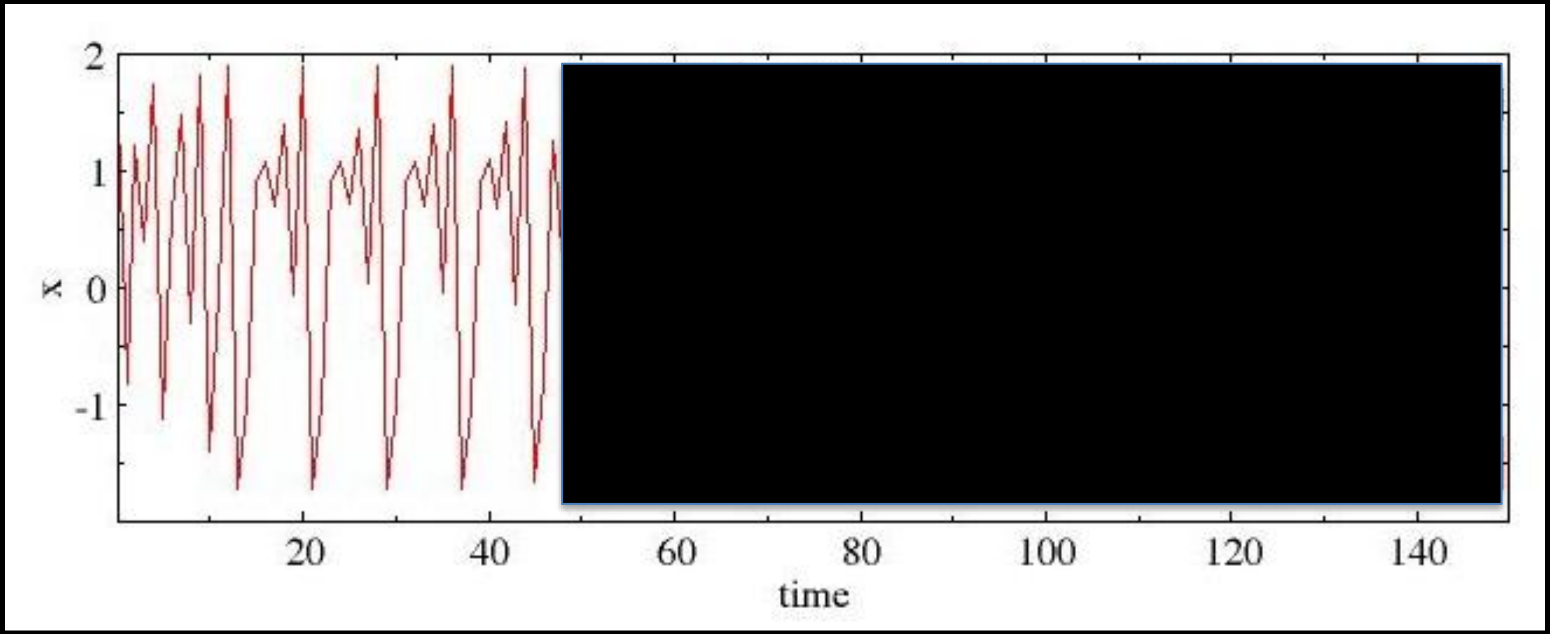
Butterfly effect: The phenomenon that a small alteration in the state of a dynamical system will cause subsequent states to differ greatly from the states that would have followed without alteration; sensitive dependence

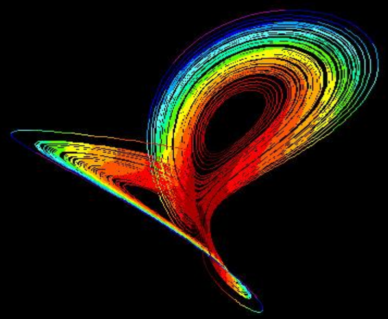
Ed Lorenz (1963): Deterministic Nonperiodic Flow

Dynamical system that is highly sensitive to perturbations of the initial conditions

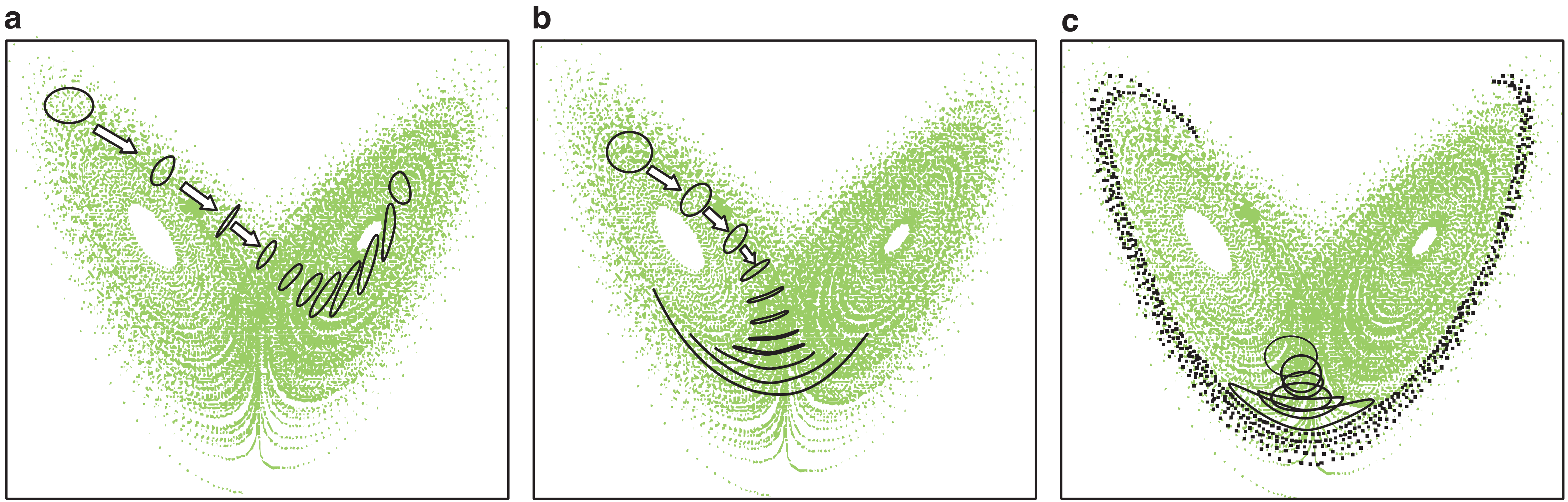
(deterministic chaos)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sigma(x-y) \\ (r-z)x-y \\ xy-bz \end{pmatrix}$$





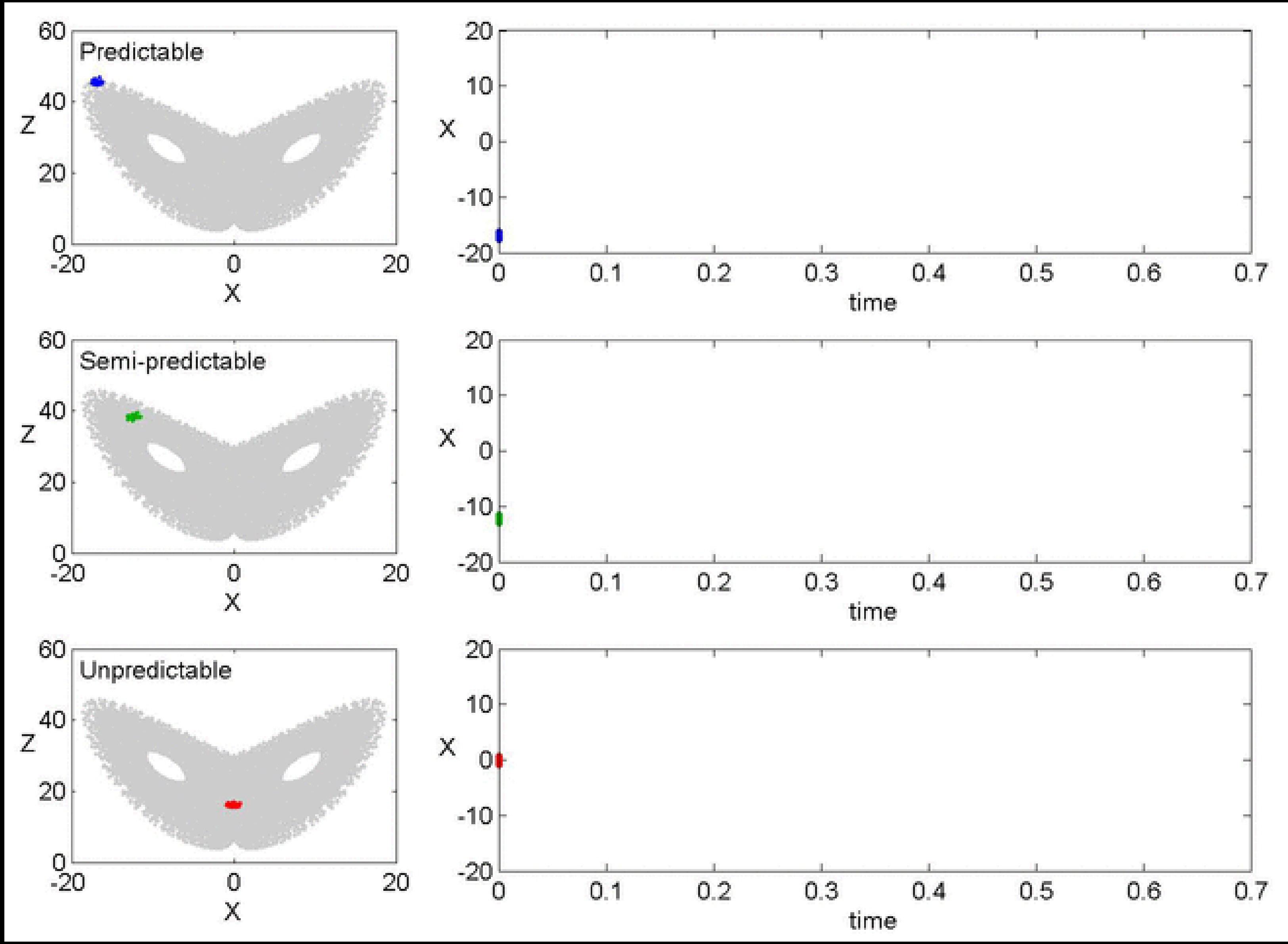
In a nonlinear system the growth of initial uncertainty is flow dependent.



predictable **semi-predictable** **unpredictable**

The set of initial conditions (black circle) is located in different regions of the attractor in a, b and c and leads to different error growth and predictability in each case.

Lorenz (1963)



Courtesy Hannah Christensen

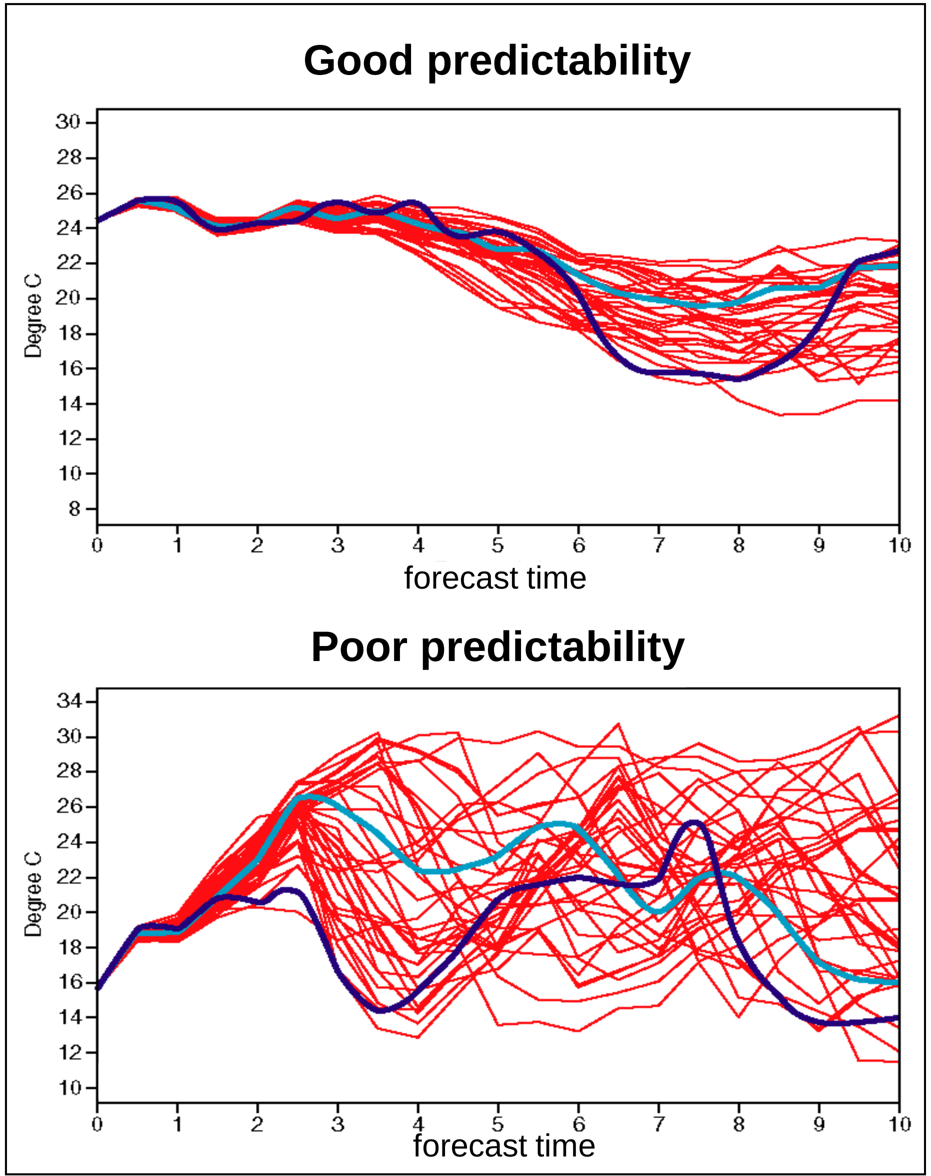
Chaos and ensemble forecasting

The atmosphere is a chaotic system where the future state of the system can be very sensitive to small differences in the current (initial) state of the system.

In practice, the initial state of the system is always uncertain due to irreducible errors in and incompleteness of observations of the initial conditions.

Our forecast models are not perfect in all aspects (e.g. small-scale features such as clouds).

Ensemble forecasting takes into account these inherent uncertainties by running a large number of similar but not identical versions of the model in parallel. The resulting forecasts are expressed in probabilities.



The "real" butterfly effect

The predictability of a flow which possesses many scales of motion

By EDWARD N. LORENZ, *Massachusetts Institute of Technology*[1]

(Manuscript received October 31, 1968, revised version December 13, 1968)

ABSTRACT

It is proposed that certain formally deterministic fluid systems which possess many scales of motion are observationally indistinguishable from indeterministic systems; specifically, that two states of the system differing initially by a small "observational error" will evolve into two states differing as greatly as randomly chosen states of the system within a finite time interval, which cannot be lengthened by reducing the amplitude of the initial error. The hypothesis is investigated with a simple mathematical model. An equation whose dependent variables are ensemble averages of the "error energy" in separate scales of motion is derived from the vorticity equation which governs two-dimensional incompressible flow. Solutions of the equation are determined by numerical integration, for cases where the horizontal extent and total energy of the system are comparable to those of the earth's atomsphere.

It is found that each scale of motion possesses an intrinsic finite range of predictability, provided that the total energy of the system does not fall off too rapidly with decreasing wave length. With the chosen values of the constants, "cumulus-scale" motions can be predicted about one hour, "synoptic-scale" motions a few days, and the largest scales a few weeks in advance. The applicability of the model to real physical systems, including the earth's atmosphere, is considered.

Introduction

The laws which govern the behavior of a fluid system—the principles of continuity of mass, momentum, and energy—are often stated

systems are indeterministic, and presumably few fluid dynamicists would question the validity of quantum mechanical principles merely because they do not customarily make use of

Lorenz (1969, Tellus)

Predictability: Does a flap of a butterfly's wings in Brazil set off a tornado in Texas?

- Talk by Ed Lorenz at a GARP session in Washington, D.C. on 29 December 1972 -

"Lest I appear frivolous in even posing the title question, let alone suggesting that it might have an affirmative answer, let me try to place it in proper perspective by offering two propositions.

1. **If a single flap of a butterfly's wings can be instrumental in generating a tornado, so also can all the previous and subsequent flaps of its wings, as can the flaps of the wings of millions of other butterflies, not to mention the activities of innumerable more powerful creatures, including our own species.**

2. **If the flap of a butterfly's wings can be instrumental in generating a tornado, it can equally well be instrumental in preventing a tornado."**

From: E. Lorenz: The Essence of Chaos (1993)

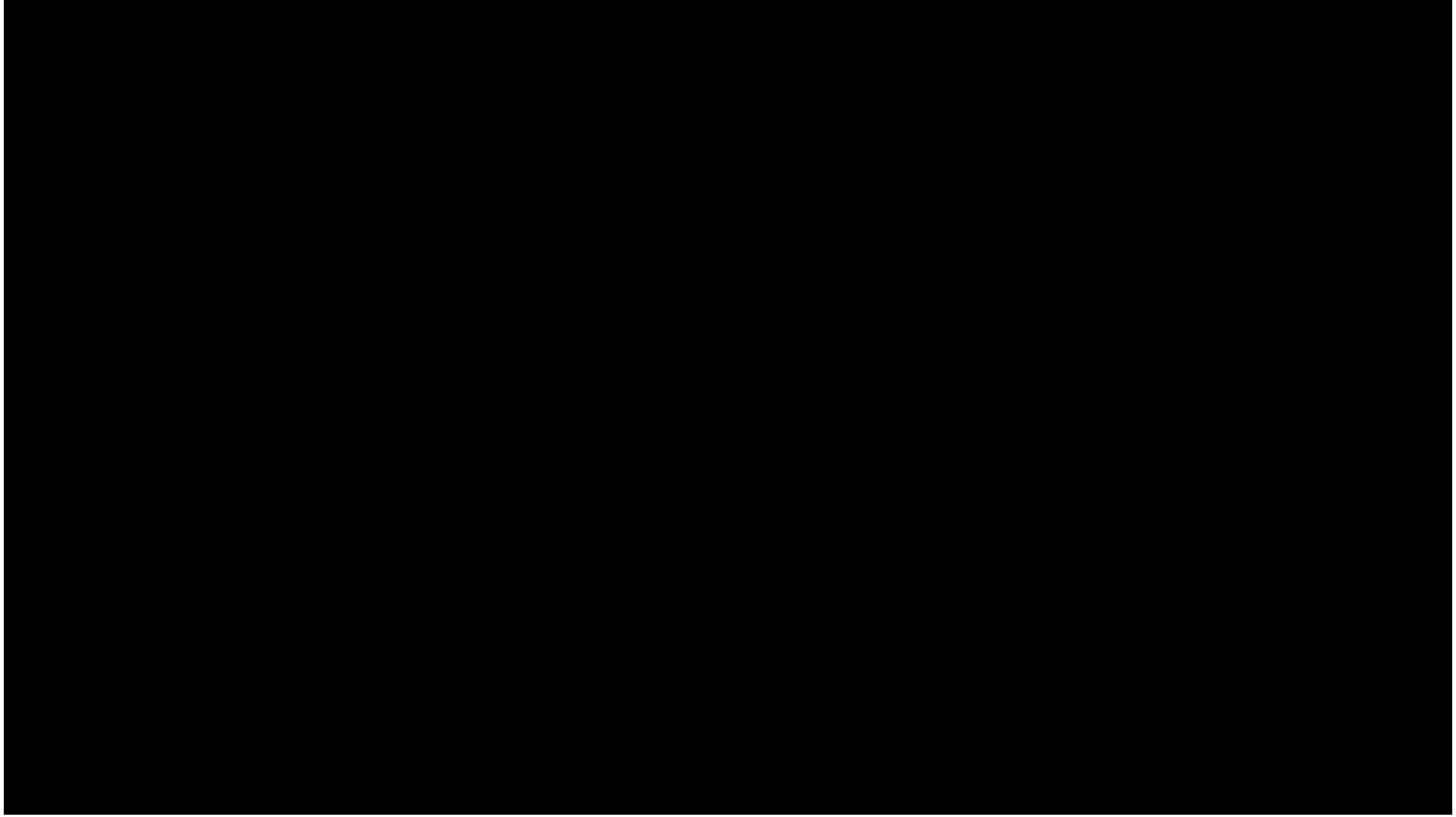
Predictability: Does a flap of a butterfly's wings in Brazil set off a tornado in Texas?

- Talk by Ed Lorenz at a GARP session in Washington, D.C. on 29 December 1972 -

The most significant results are the following.

1. **Small errors in the coarser structure of the weather pattern-** those features which are readily resolved by conventional observing networks - **tend to double in about 3 days. As the errors become larger the growth rate subsides.** This limitation alone would allow us to extend the range of acceptable prediction by 3 days every time we cut the observation error in half, and would offer the hope of eventually making good forecasts several weeks in advance.
2. **Small errors in the finer structure** - e.g., the positions of individual clouds - **tend to grow much more rapidly, doubling in hours or less.** This limitation alone would not seriously reduce our hopes for extended-range forecasting, since ordinarily we do not forecast the finer structure at all.
3. **Errors in the finer structure, having attained appreciable size, tend to induce errors in the coarser structure.** This result, which is less firmly established than the previous ones, implies that after a day or so there will be appreciable errors in the coarser structure, which will thereafter grow just as if they had been present initially. **Cutting the observation error in the finer structure in half would extend the range of acceptable prediction of even the coarser structure only by hours or less**. The hopes for predicting two weeks or more in advance are thus greatly diminished.
4. **Certain special quantities such as weekly average temperatures and weekly total rainfall may be predictable at a range at which entire weather patterns are not**.

From: E. Lorenz: The Essence of Chaos (1993)



https://youtu.be/bZ6yxt_o_CQ?feature=shared