Introduction to element based computing --- finite volume and finite element methods. Mesh generation

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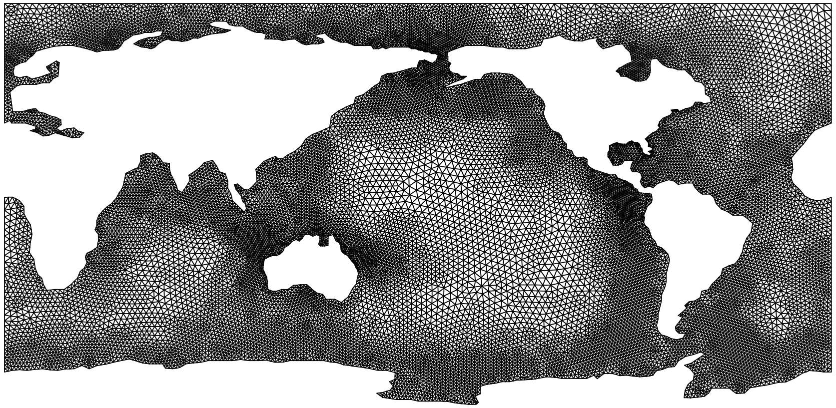
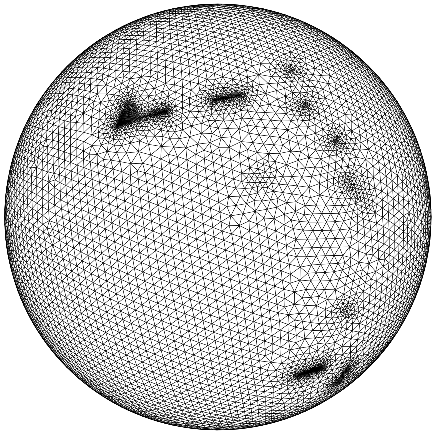
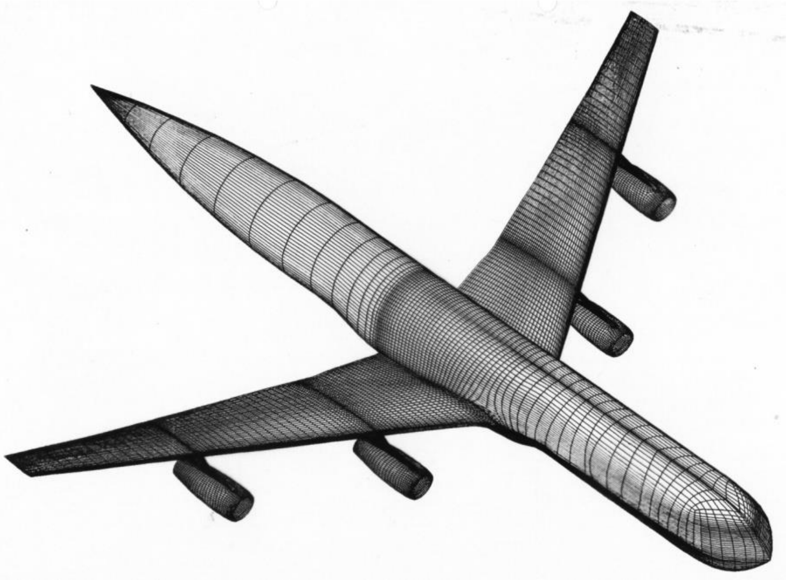


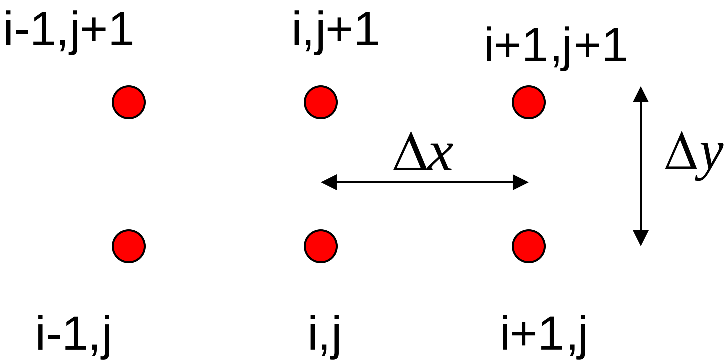


Traditional Discretisation Methods

- Finite Difference
- Finite Element
- Finite Volume



Finite Difference Method



$$\frac{\partial (u\rho)}{\partial x}\Big|_{i,j} \approx \frac{((u\rho)_{i+1,j} - (u\rho)_{i-1,j})}{2\Delta x} \qquad \boxed{O(\Delta x^2)}$$

central

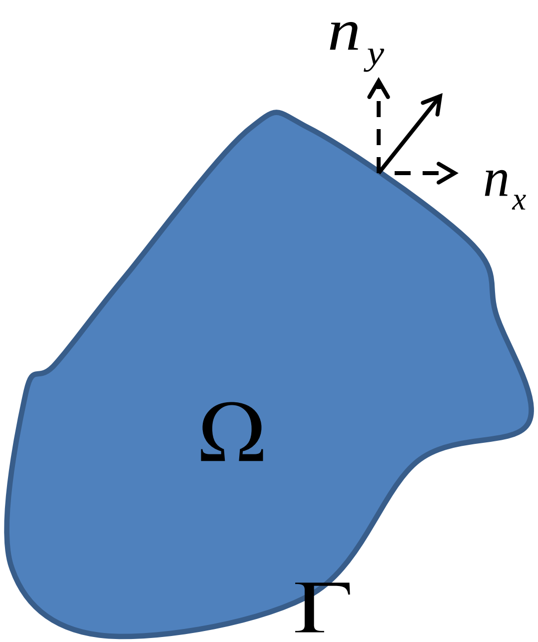
$$\frac{\partial (u\rho)}{\partial x}\Big|_{i,j} \approx \frac{((u\rho)_{i+1,j} - (u\rho)_{i,j})}{\Delta x} \qquad \boxed{O(\Delta x)}$$

forward

$$\frac{\partial \rho}{\partial t} + \frac{\partial (u\rho)}{\partial x} + \frac{\partial (v\rho)}{\partial y} = 0$$

$$\frac{\partial}{\partial t}\rho_{i,j} + \frac{((u\rho)_{i+1,j} - (u\rho)_{i-1,j})}{2\Delta x} + \frac{((v\rho)_{i,j+1} - (v\rho)_{i,j-1})}{2\Delta y} = 0$$

Finite Volume Method



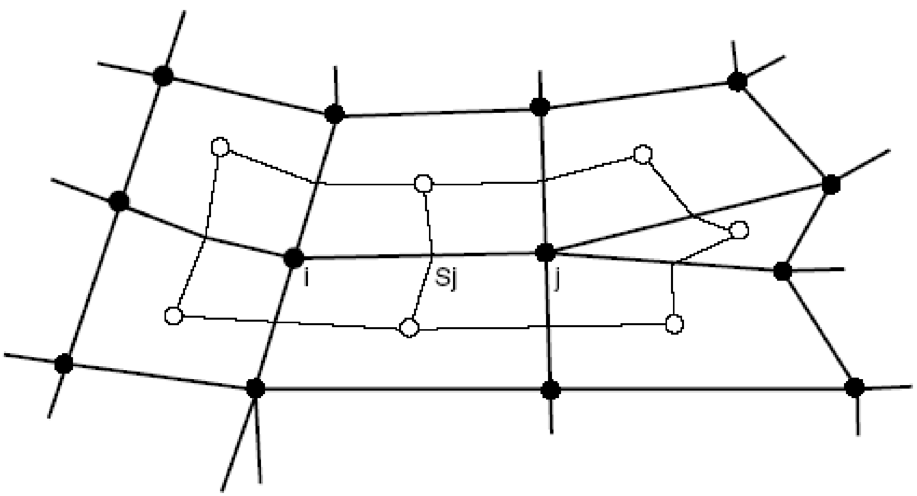
$$\frac{\partial \rho}{\partial t} + \frac{\partial (u\rho)}{\partial x} + \frac{\partial (v\rho)}{\partial y} = 0$$

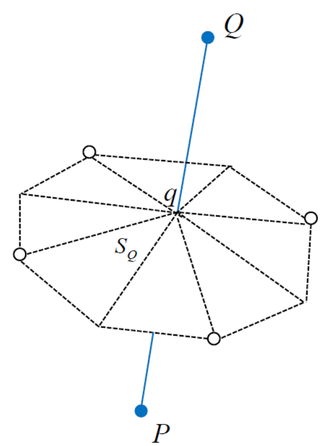
$$\int_\Omega \frac{\partial \rho}{\partial t} d\Omega + \int_\Omega \frac{\partial (u\rho)}{\partial x} d\Omega + \int_\Omega \frac{\partial (v\rho)}{\partial y} d\Omega = 0$$

From Gauss Divergence Theorem:

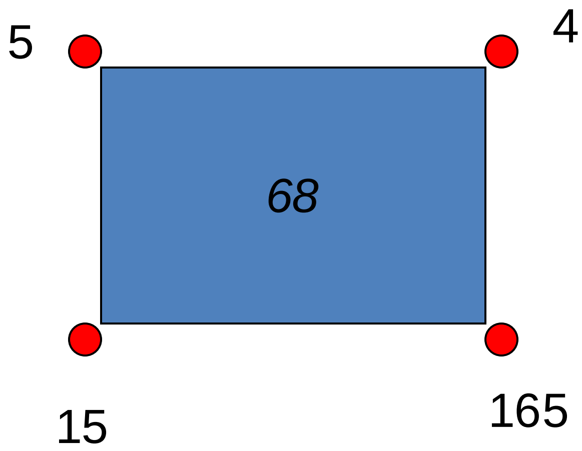
$$\frac{\partial}{\partial t}\int_\Omega \rho d\Omega + \int_\Gamma (u\rho) n_x d\Gamma + \int_\Gamma (v\rho) n_y d\Gamma = 0$$

$$\frac{\partial}{\partial t}\rho_i V_i + \sum_j (u\rho)_{ij}\, S_x + \sum_i (v\rho)_{ij}\, S_y = 0$$





Finite Element Method



Element	Nodes
68	5 4 165 15

(Repeated index notation is used here)

$$\rho \approx \rho_i N_i =$$

$$\rho_5 N_5 + \rho_4 N_4 + \rho_{165} N_{165} + \rho_{15} N_{15}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (u\rho)}{\partial x} + \frac{\partial (v\rho)}{\partial y} = 0$$

$$\int_\Omega \frac{\partial \rho}{\partial t} d\Omega + \int_\Omega \frac{\partial (u\rho)}{\partial x} d\Omega + \int_\Omega \frac{\partial (v\rho)}{\partial y} d\Omega = 0$$

$$\frac{\partial}{\partial t} \int_\Omega \rho_i N_i d\Omega + \int_\Omega \frac{\partial (u\rho)_i N_i}{\partial x} d\Omega + \int_\Omega \frac{\partial (v\rho)_i N_i}{\partial y} d\Omega = 0$$

Finite Element Method continued

Weighted residual analysis:

$$\frac{\partial}{\partial t}\int_{\Omega}\rho_i N_i W_j d\Omega+\int_{\Omega}\frac{\partial(u\rho)_i N_i}{\partial x}W_j d\Omega+\int_{\Omega}\frac{\partial(v\rho)_i N_i}{\partial y}W_j d\Omega=0$$

$$\frac{\partial}{\partial t}(\int_{\Omega}N_i W_j d\Omega)\rho_i+(\int_{\Omega}\frac{\partial(N_i)}{\partial x}W_j d\Omega)u\rho_i+(\int_{\Omega}\frac{\partial(N_i)}{\partial y}W_j d\Omega)v\rho_i=0$$

If W is chosen to be the same as N the method is called Galerkin method.

$$\frac{\partial}{\partial t}(\int_{\Omega}N_i N_j d\Omega)\rho_i+(\int_{\Omega}\frac{\partial(N_i)}{\partial x}N_j d\Omega)u\rho_i+(\int_{\Omega}\frac{\partial(N_i)}{\partial y}N_j d\Omega)v\rho_i=0$$

For easy implementation of boundary conditions this is integrated by parts.

$$\frac{\partial}{\partial t}(\int_{\Omega}N_i N_j d\Omega)\rho_i-(\int_{\Omega}N_i\frac{\partial(N_j)}{\partial x}d\Omega)u\rho_i-(\int_{\Omega}N_i\frac{\partial(N_j)}{\partial y}d\Omega)v\rho_i+(\int_{\Gamma}N_i N_j n_x d\Gamma)u\rho_i+(\int_{\Gamma}N_i N_j n_y d\Gamma)v\rho_i=0$$

$$\frac{\partial}{\partial t}\mathbf{M}_{elem}\rho_i+\mathbf{B}_{Xelem}u\rho_i+\mathbf{B}_{Yelem}v\rho_i=0$$

Finite Element Method

1) Divides computational space into elements
$$\frac{\partial}{\partial t}\mathbf{M}_{elem}\rho_i + \mathbf{B}_{Xelem}u\rho_i + \mathbf{B}_{Yelem}v\rho_i = 0$$

For every element constructs
$$\mathbf{M}_{elem}, \mathbf{B}_{Xelem}, \mathbf{B}_{Yelem}; \boldsymbol{\rho}_{elem}, \mathbf{u}\boldsymbol{\rho}_{elem}, \mathbf{v}\boldsymbol{\rho}_{elem}$$

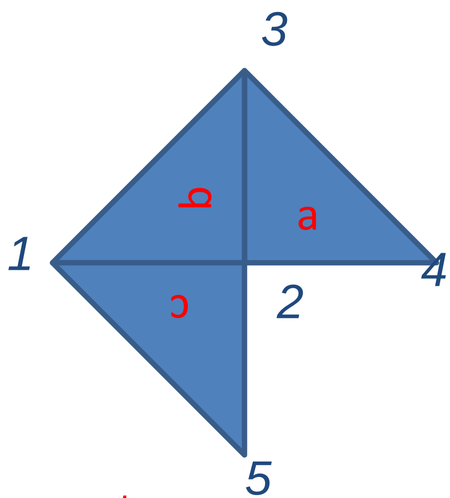
2) Reconnects elements at nodes

Agglomerates for the whole domain
$$\mathbf{M} = \sum_e \mathbf{M}_{elem}, \mathbf{B}_X = \sum_e \mathbf{B}_{Xelem}, \mathbf{B}_Y = \sum_e \mathbf{B}_{Yelem}$$
$$\boldsymbol{\rho} = \sum_e \boldsymbol{\rho}_{elem}, \mathbf{u}\boldsymbol{\rho} = \sum_e \mathbf{u}\boldsymbol{\rho}_{elem}, \mathbf{v}\boldsymbol{\rho} = \sum_e \mathbf{v}\boldsymbol{\rho}_{elem}$$

3) As a result a set of algebraic equations is formed and its solution follows

$$\frac{\partial}{\partial t}\mathbf{M}\,\boldsymbol{\rho} + \mathbf{B}_X\mathbf{u}\boldsymbol{\rho} + \mathbf{B}_Y\mathbf{v}\boldsymbol{\rho} = \mathbf{0}$$

Matrix Agglomeration



Element a

$$\begin{Bmatrix}\begin{bmatrix} a22 & a23 & a24 \\ a32 & a33 & a34 \\ a42 & a43 & a44 \end{bmatrix}\begin{Bmatrix} \rho 2 \\ \rho 3 \\ \rho 4 \end{Bmatrix}\end{Bmatrix}$$

Element b

$$\begin{Bmatrix}\begin{bmatrix} b11 & b12 & b13 \\ b21 & b22 & b23 \\ b31 & b32 & b33 \end{bmatrix}\begin{Bmatrix} \rho 1 \\ \rho 2 \\ \rho 3 \end{Bmatrix}\end{Bmatrix}$$

Element c

$$\begin{Bmatrix}\begin{bmatrix} c11 & c12 & c15 \\ c21 & c22 & c25 \\ c51 & c52 & c55 \end{bmatrix}\begin{Bmatrix} \rho 1 \\ \rho 2 \\ \rho 5 \end{Bmatrix}\end{Bmatrix}$$

Global matrix is a sum of all element matrixes

$$\begin{Bmatrix}\begin{bmatrix} b11+c11 & b12+c12 & b13 & 0 & c15 \\ b21+c21 & a22+b22+c22 & a23+b23 & a24 & c25 \\ b31 & a32+b32 & a33+b33 & a34 & 0 \\ 0 & a42 & a43 & a44 & 0 \\ c51 & c52 & 0 & 0 & c55 \end{bmatrix}\begin{Bmatrix} \rho 1 \\ \rho 2 \\ \rho 3 \\ \rho 4 \\ \rho 5 \end{Bmatrix}\end{Bmatrix}$$

$$\frac{\partial}{\partial t}\mathbf{M}\ \boldsymbol{\rho} + \mathbf{B}_X \mathbf{u}\boldsymbol{\rho} + \mathbf{B}_Y \mathbf{v}\boldsymbol{\rho} = \mathbf{0}$$

e.g. For the linear triangular element the consistent mass matrix

$$\begin{Bmatrix}\begin{bmatrix} a22 & a23 & a24 \\ a32 & a33 & a34 \\ a42 & a43 & a44 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\end{Bmatrix} \cdot \ \textit{Area Element"a"} \ /12$$

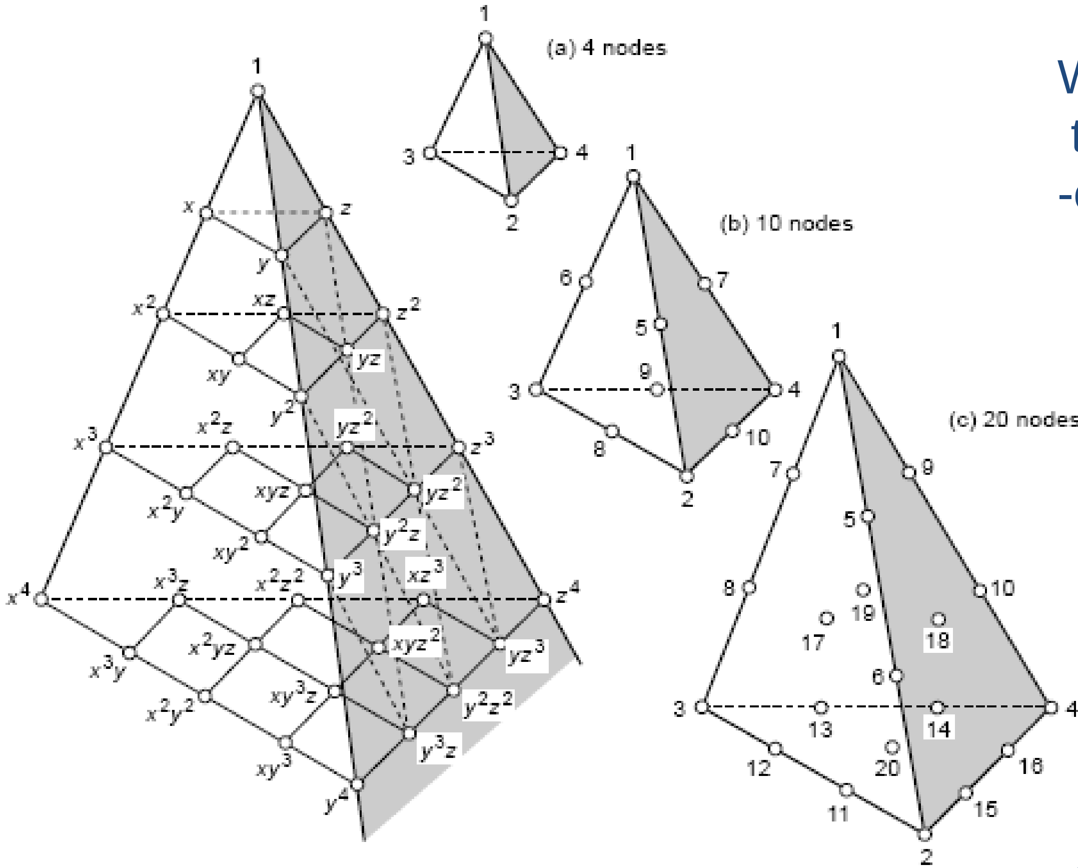
SHAPE FUNCTIONS

$$\rho \approx \sum_i N_i \, \rho_i$$

And when derivatives are of interest:

$$\frac{d\rho}{dx} \approx \sum_i \frac{dN_i}{dx} \rho_i$$

The tetrahedron family of elements



We know functions N,
they are frequently polynomials
-obtaining their derivatives is easy.

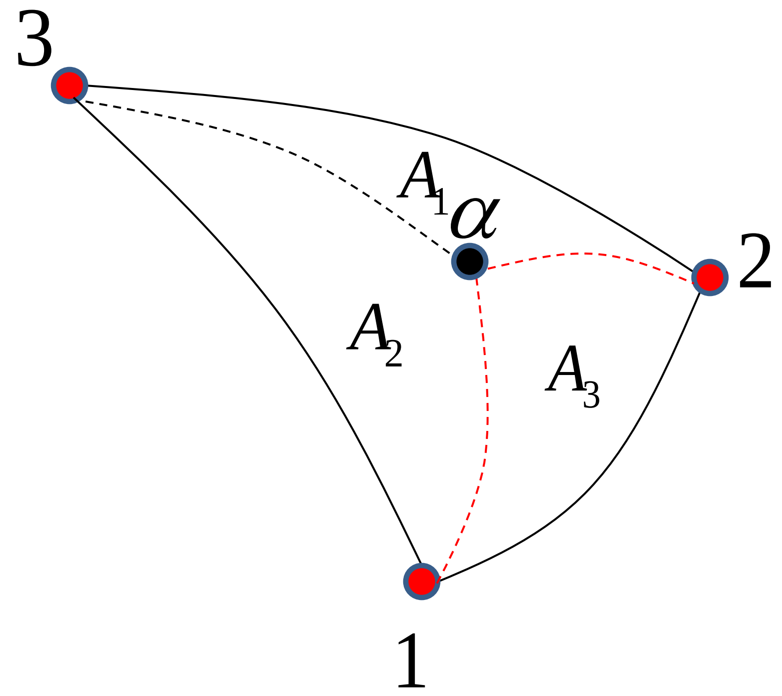
After Zienkiewicz et al FEM 2000

SHAPE FUNCTIONS

Example: a linear interpolation of a scalar T in a triangle. The value of T in an arbitrary point alpha is approximated by:

$$\rho_\alpha \approx \rho_1 N_1 + \rho_2 N_2 + \rho_3 N_3$$

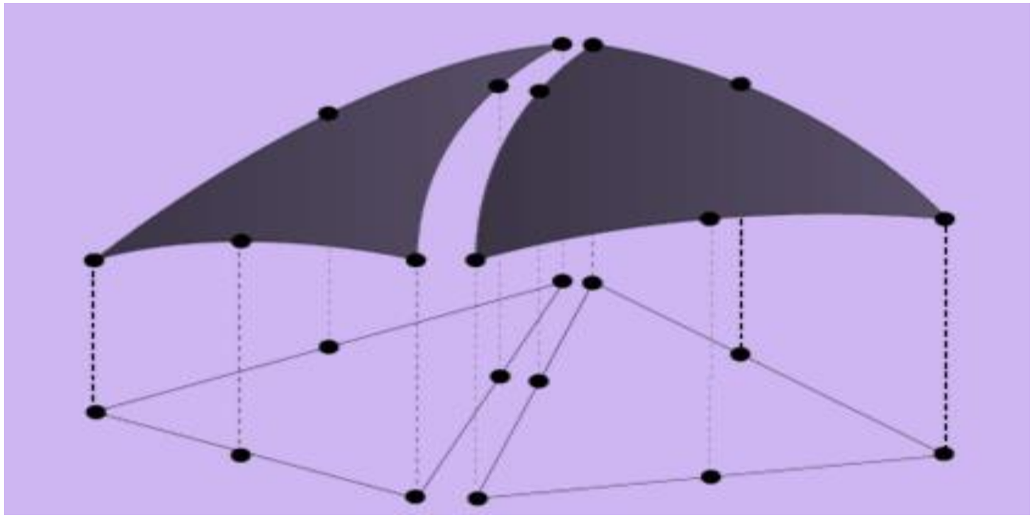
$$N_1 = A_1 / A_{element}$$



where A is an area

SHAPE FUNCTIONS

Curvilinear elements can be formed using transformations

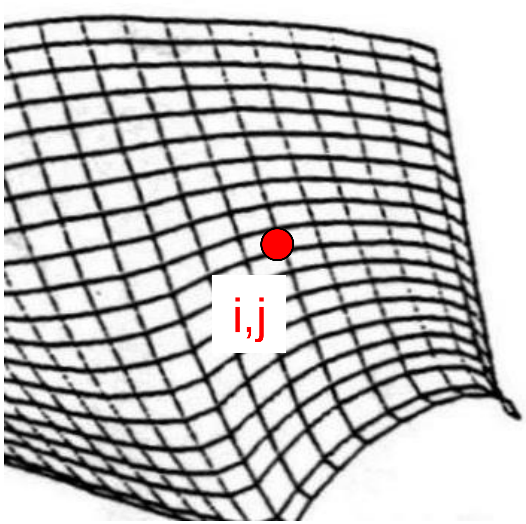


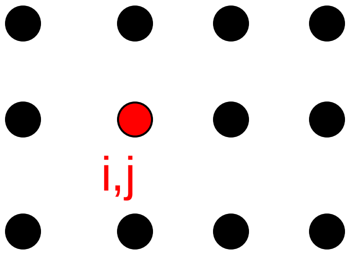
Isoparametric elements

Typical Examples of Data Structures

Structured

Point based --- I,J,K indexing
Set of coordinates and connectivities
Naturally map into the elements of a matrix

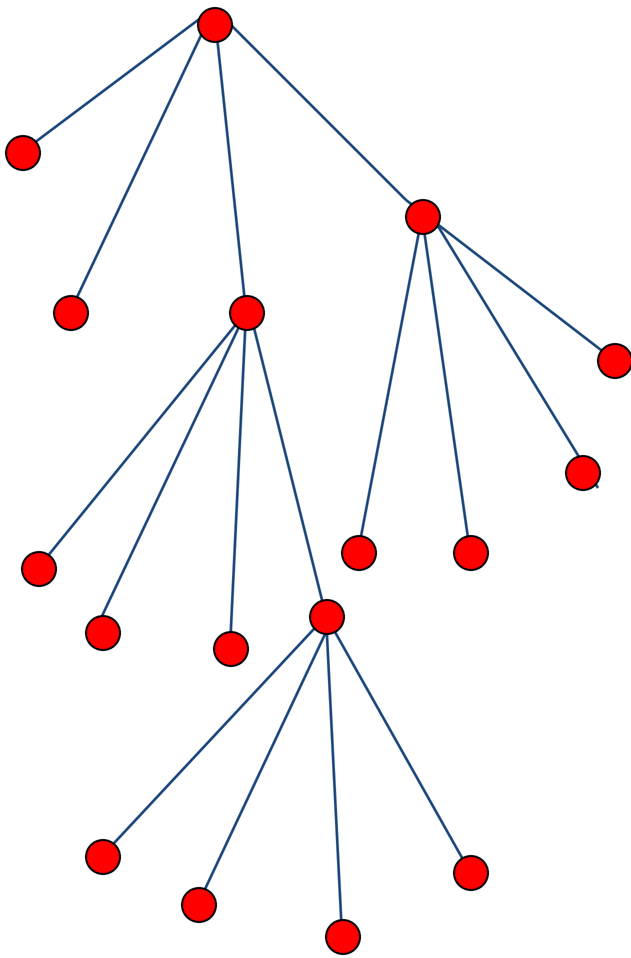


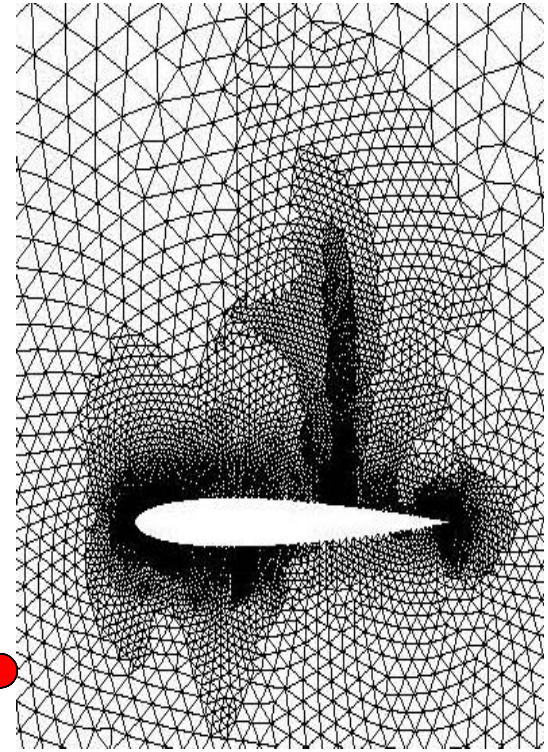


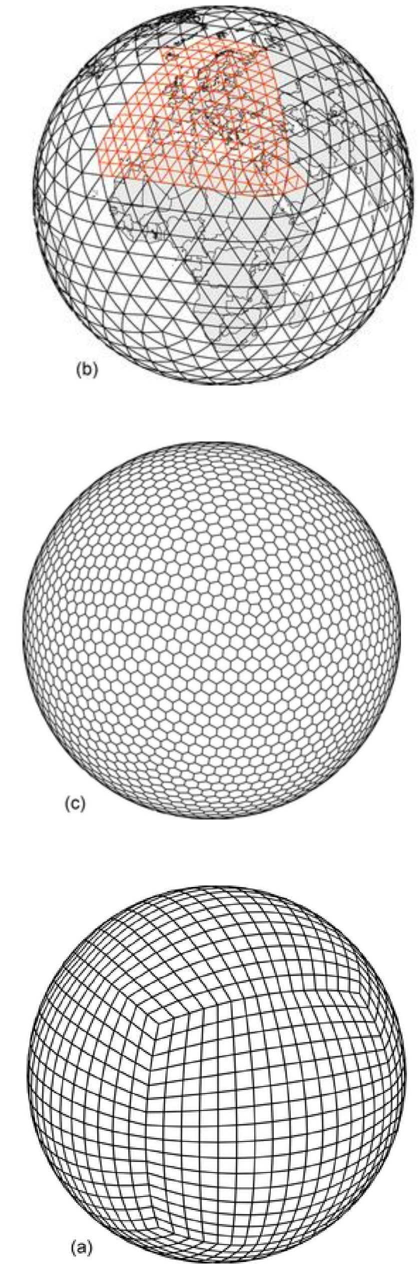
Typical Examples of Data Structures

Structured

Binary trees





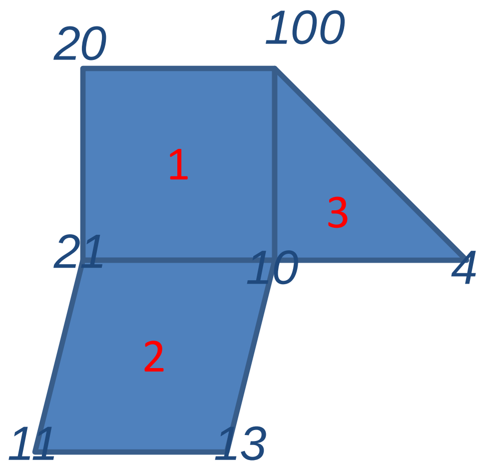


Source: Ullrich et al GMD 2017

Typical Examples of Data Structures

Unstructured

Element based connectivity

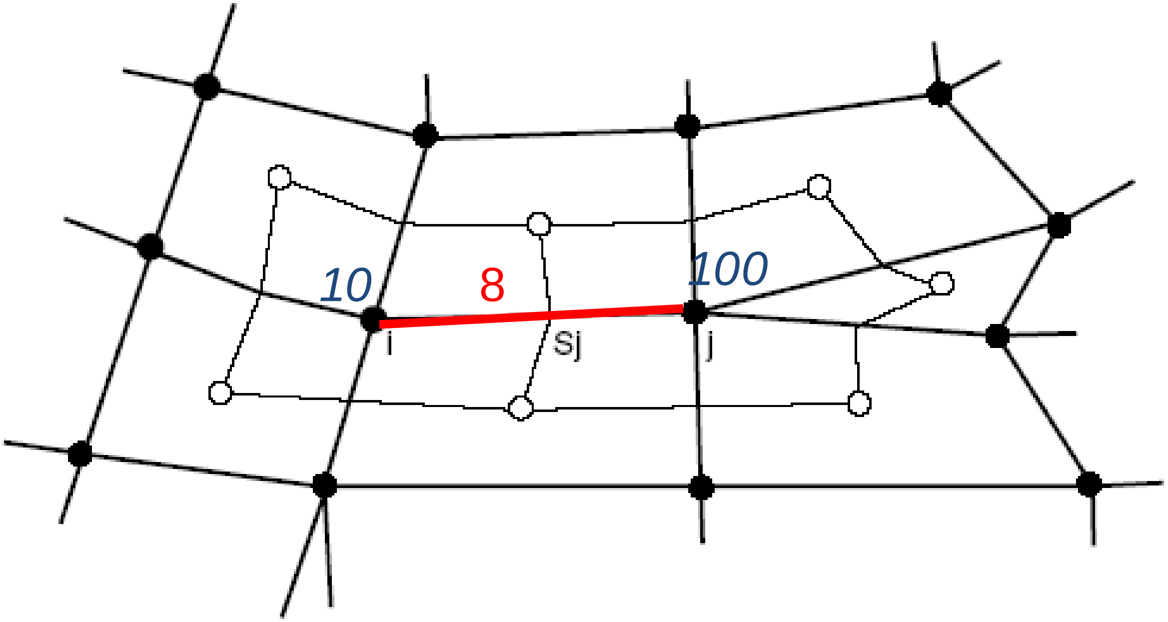


Element 1 10 100 20 21
Element 2 21 11 13 10
Element 3 4 100 10

+ information related to shape functions

Typical Examples of Data Structures

Unstructured



Edge 8 10 100

+ geometrical information

Edge based data

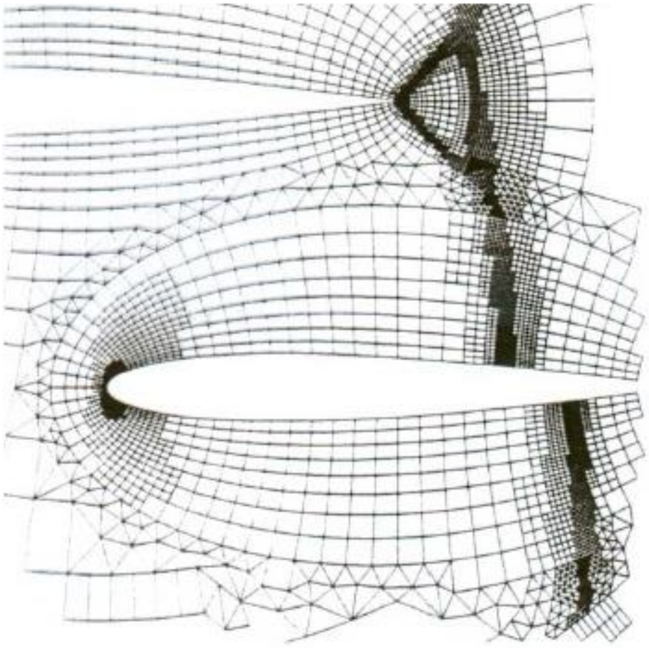
☺ *Flexible mesh adaptivity and hybrid meshes*

☺ *Low storage*

☺ *Easy generalisation to 3D,*

☺ *Less expensive than element based data structure*

More expensive operations than I,J,K



Selected Mesh Generation Techniques

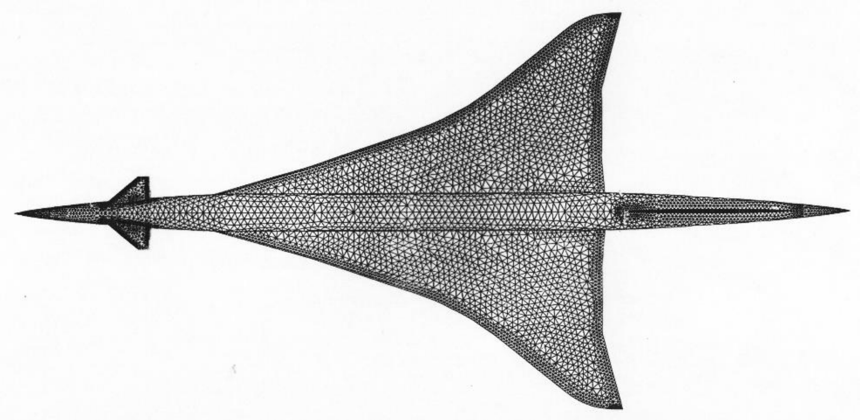
Unstructured Meshes

Direct triangulation

Advancing Front Technique

Delaunay Triangulation

Others

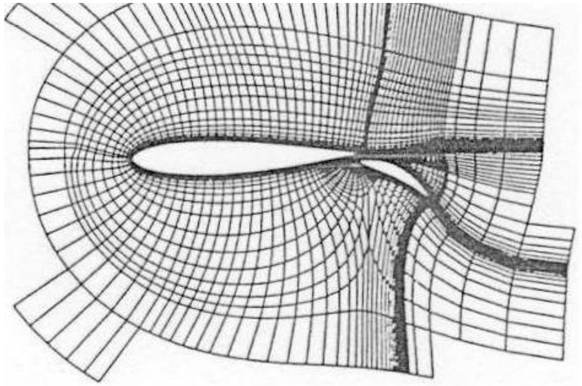


Structured Meshes

Cartesian grids with mapping and/or immersed boundaries

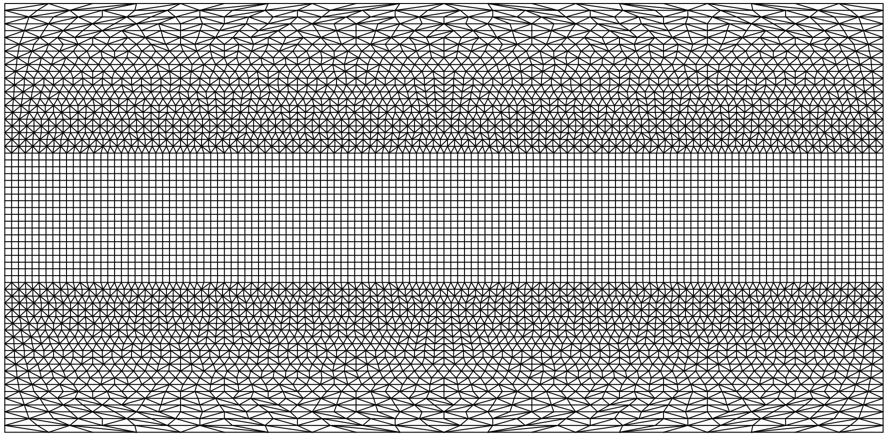
Variants of icosahedral meshes

Others

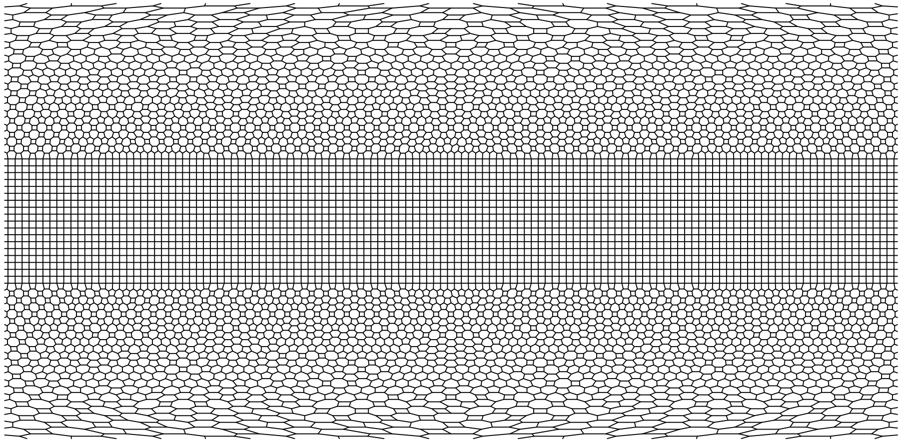


An example of the direct triangulation

Reduced Gaussian Grid

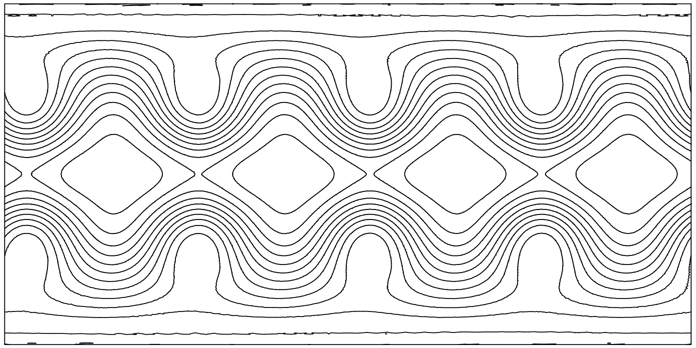


Primary mesh

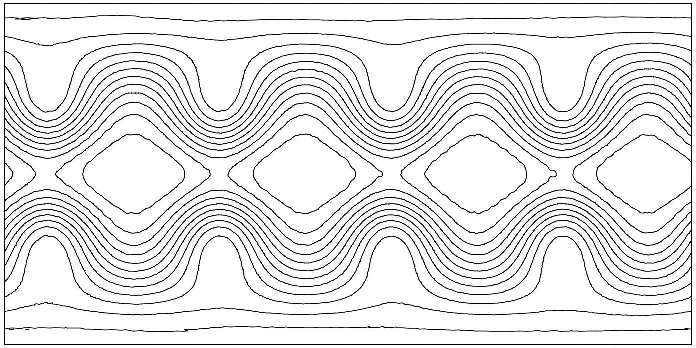


Dual mesh

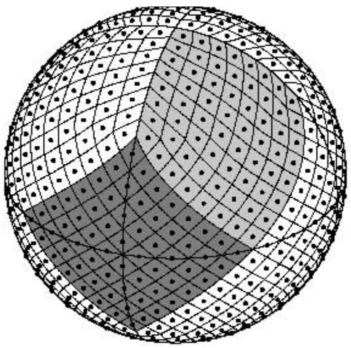
Rossby-Haurwitz Wave



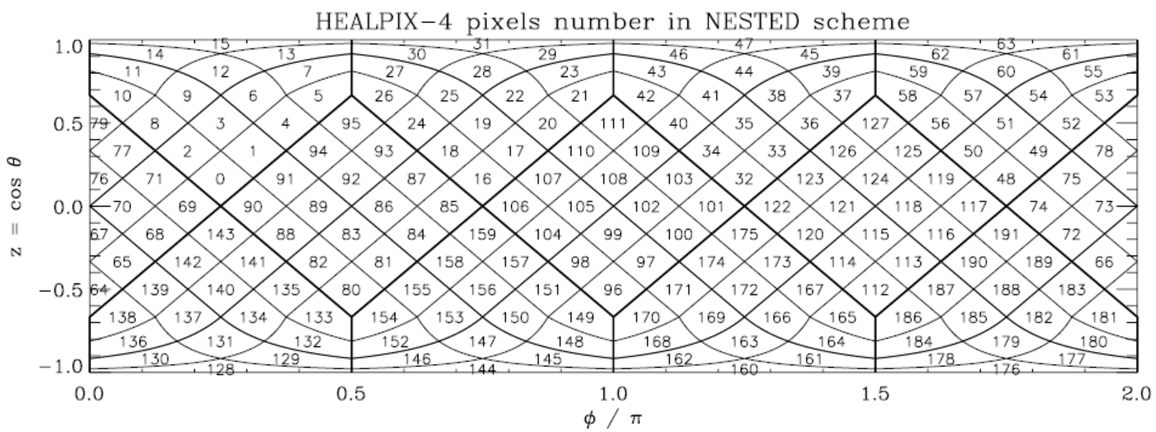
5 days



14 days

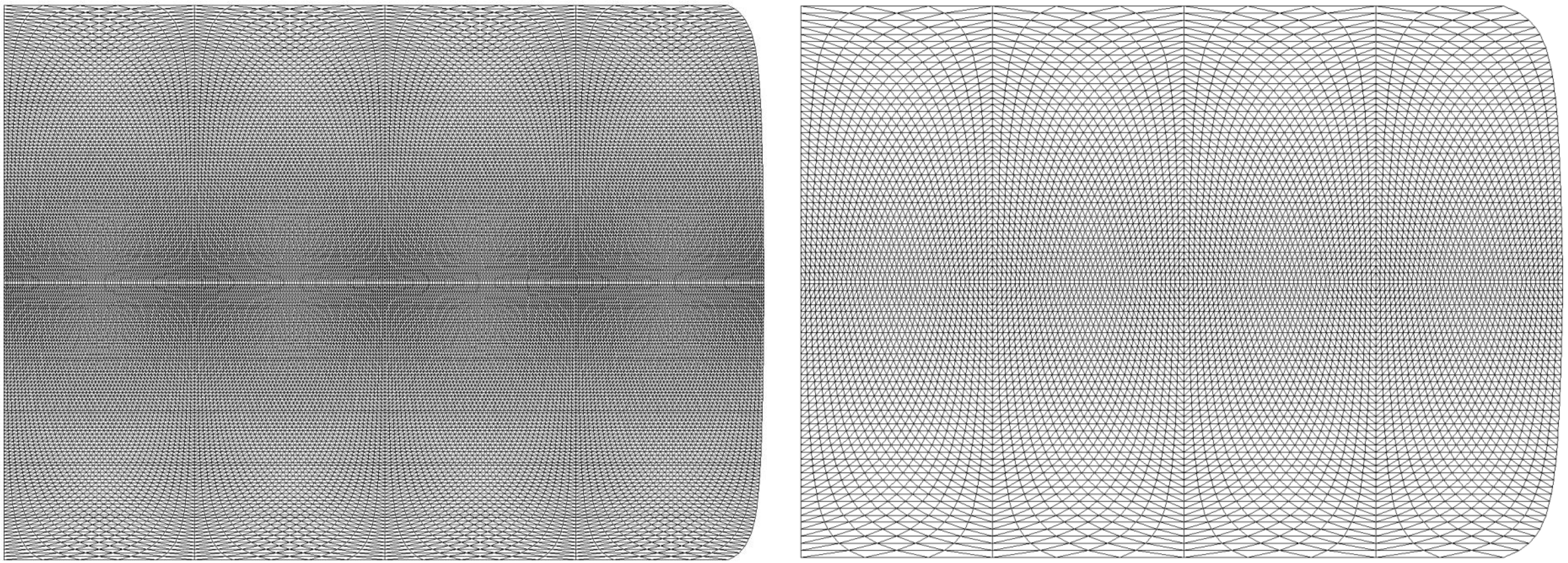


An example of bespoke mesh



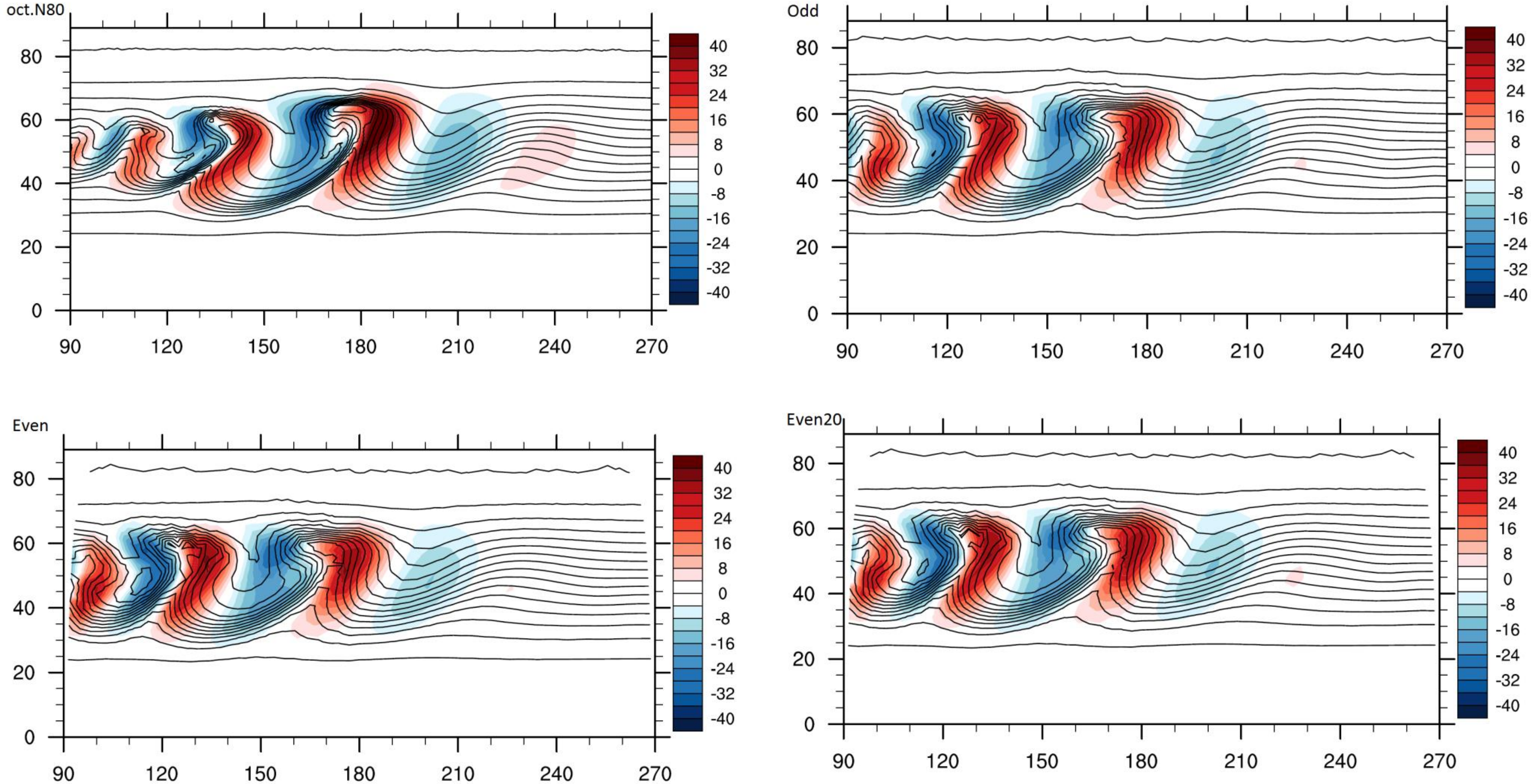
Source: Gorski et al
Astrophysical Journal 2005

Oct.N80 and reduced meshes



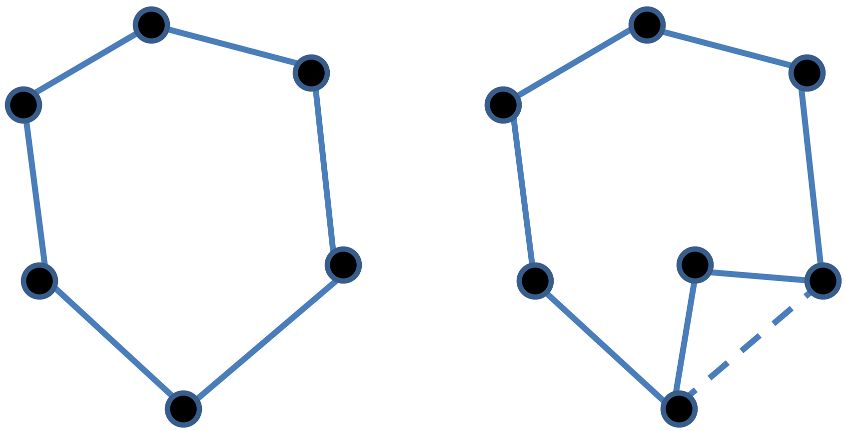
Oct.N80 fine and ‘odd’ reduced meshes

Reduced Mesh – Baroclinic Instability

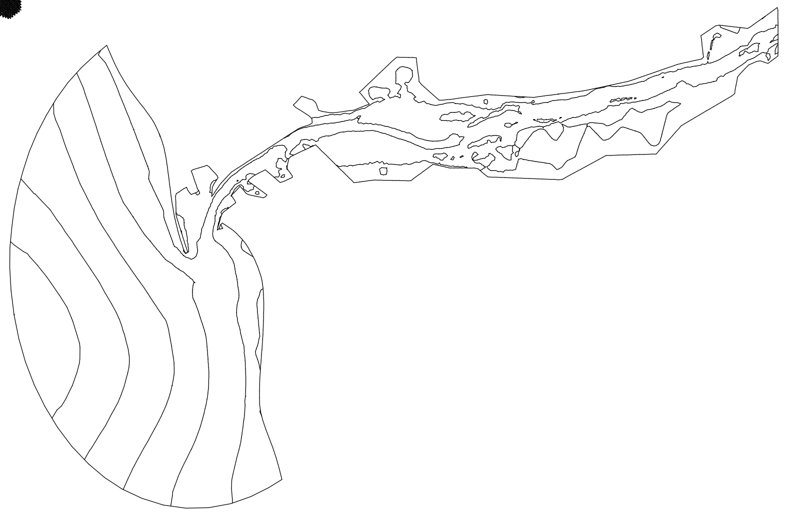


Day 8, horizontal velocity and potential temperature

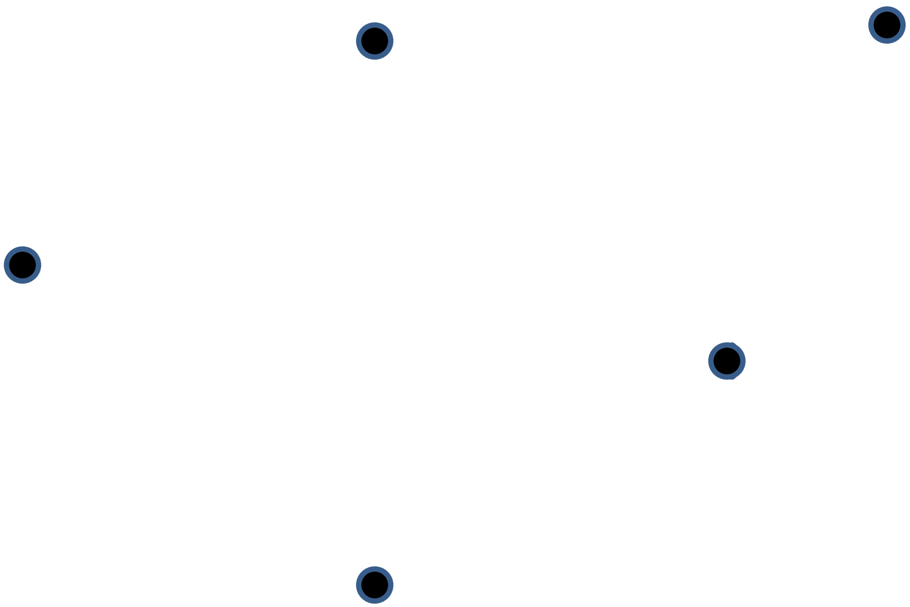
Advancing Front Technique



Simultaneous mesh point
generation and connectivity

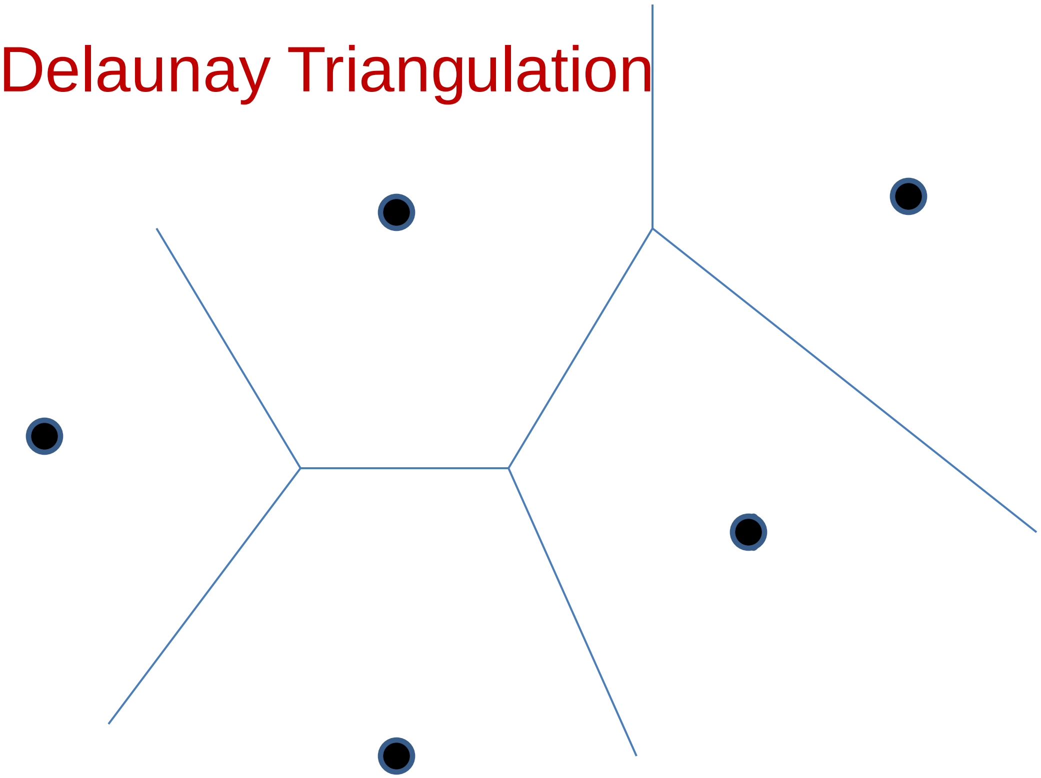


Delaunay Triangulation



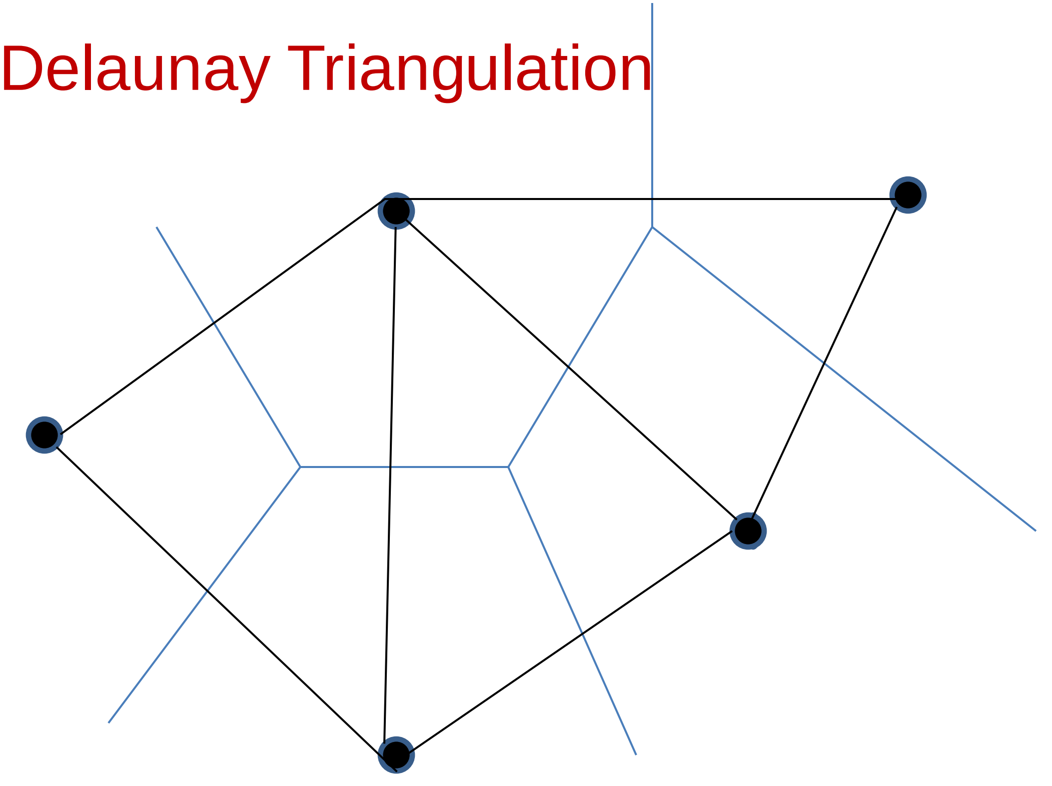
How to connect a given set of points?

Delaunay Triangulation



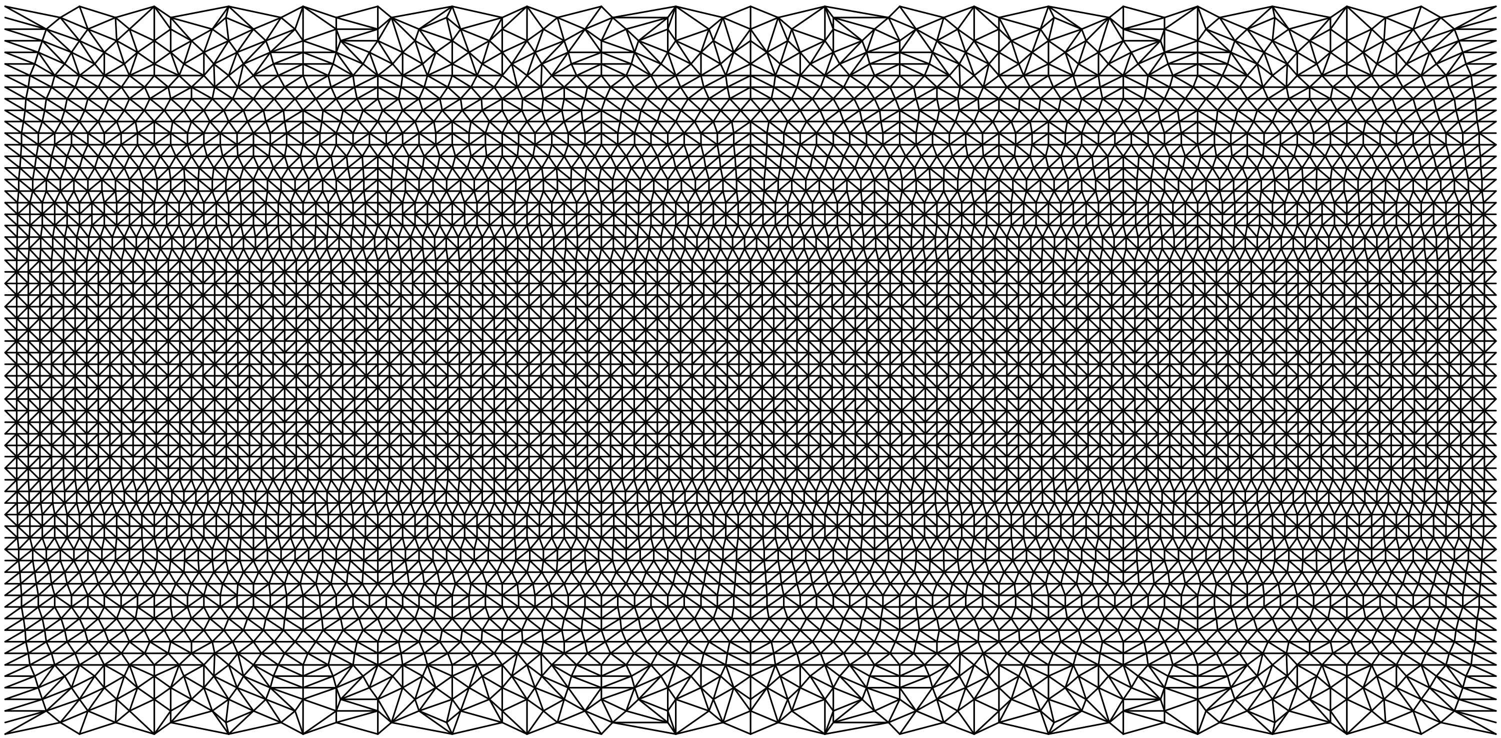
Create Voronoi polygons, i.e.
The construct that assigns to each point the area
of the plane closer to that point than to any other point
in the set. A side of a Voronoi polygon must be midway between the two points
which it separates

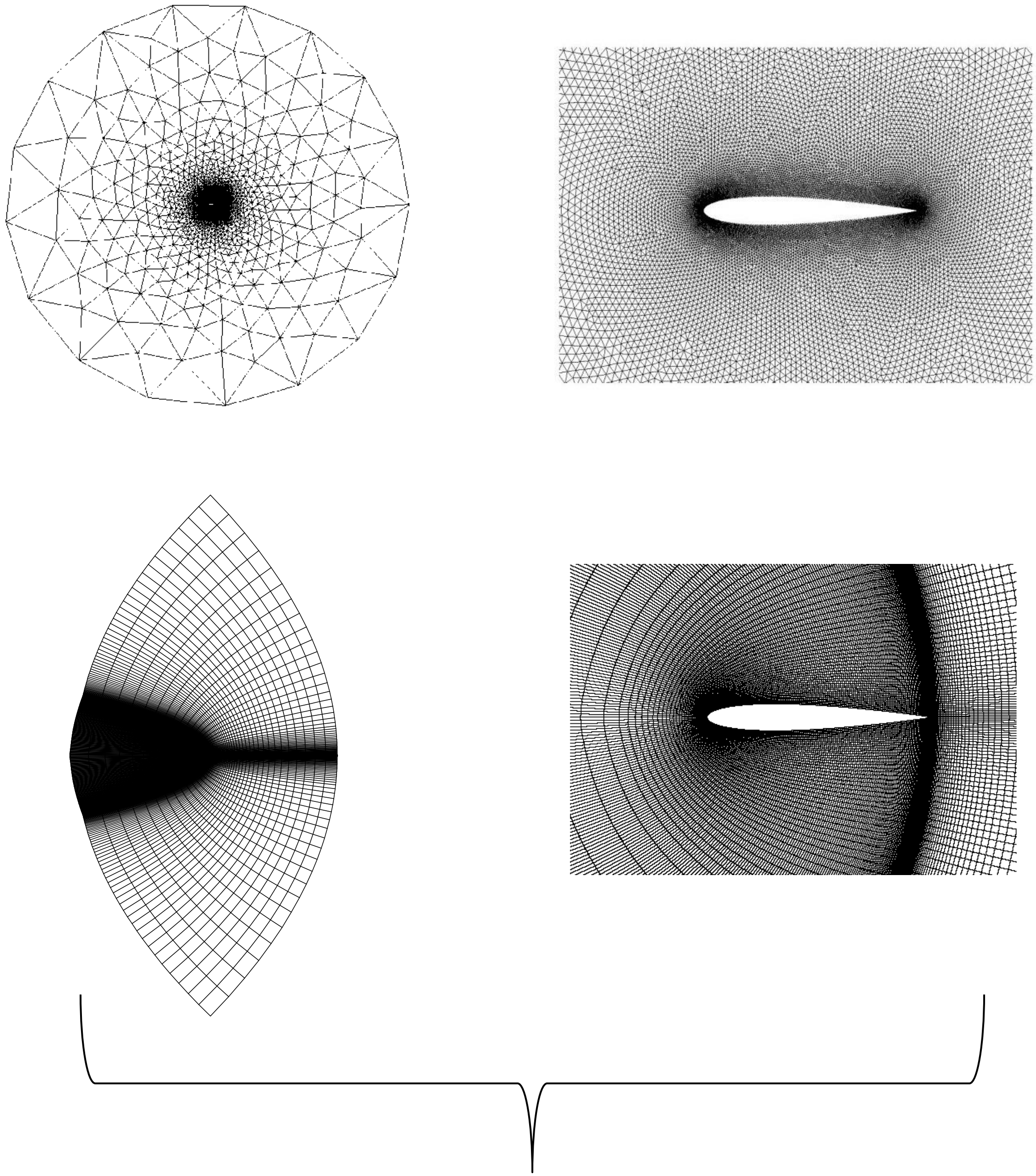
Delaunay Triangulation



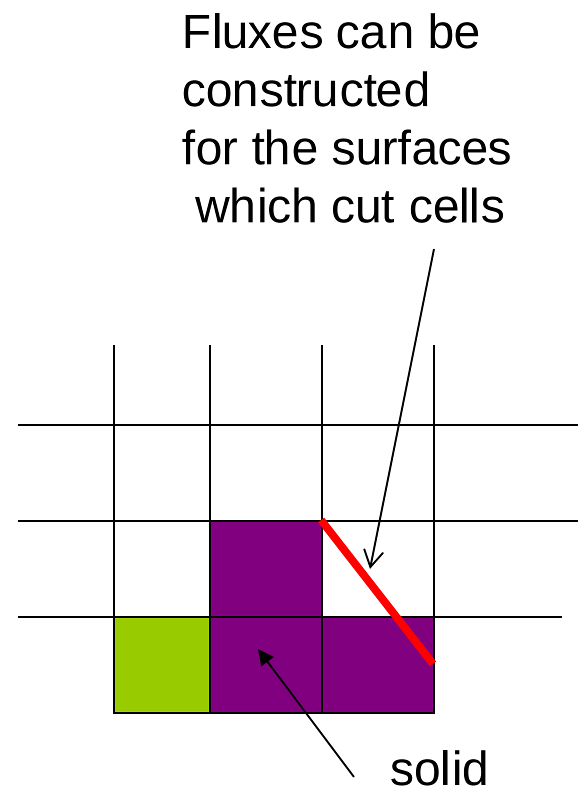
If all point pairs of which have some boundary in common are joint by straight lines, the result is a triangulation of the convex hull of the points.

Delaunay Triangulation mesh constructed from the reduced Gaussian grid points





Geometry conforming meshes

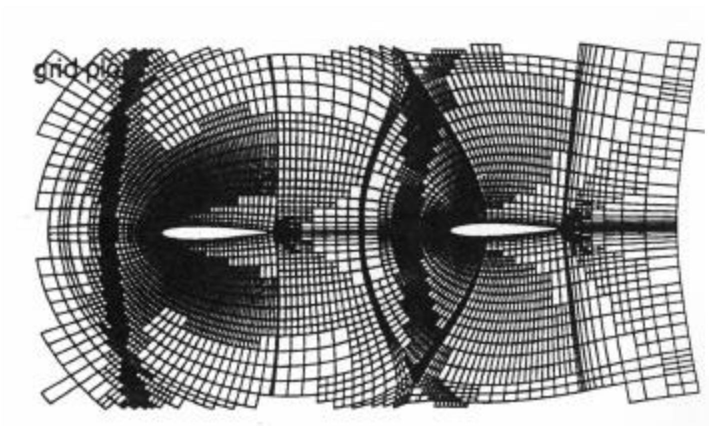


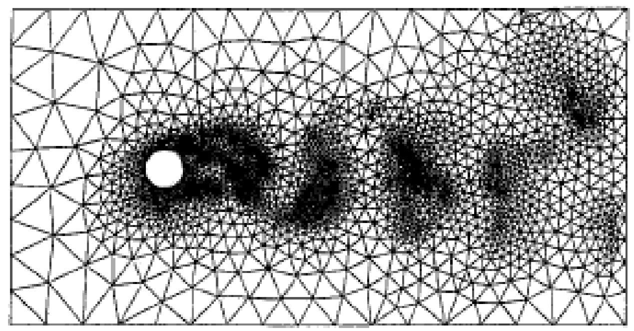
Meshing techniques for mesh adaptivity

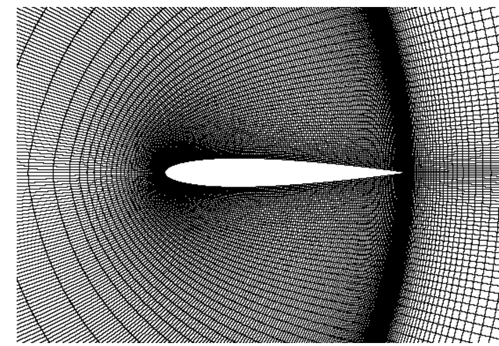
-for lower order elements are:

point enrichment (h-refinement),

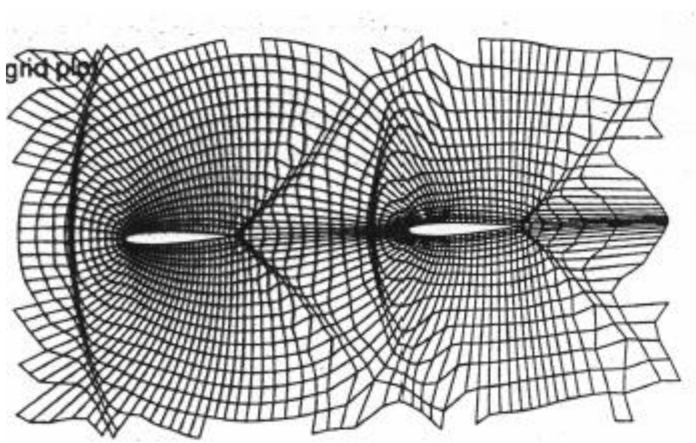
and automatic regeneration

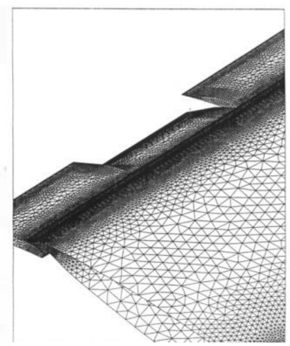


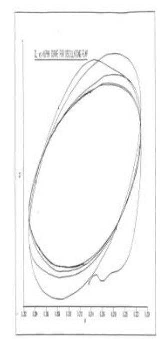




mesh movement



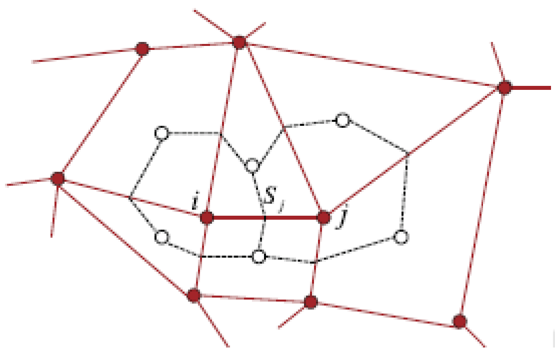


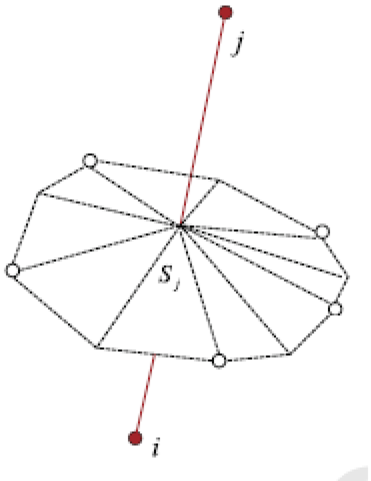


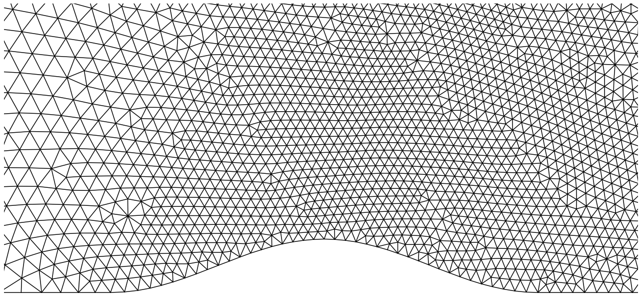
Dynamic regenerations for Meshes utilising a mapping are particularly efficient

-for higher order elements
(p-refinement) increases order of shape function that is the order of **p**olynomial used for elemental interpolation

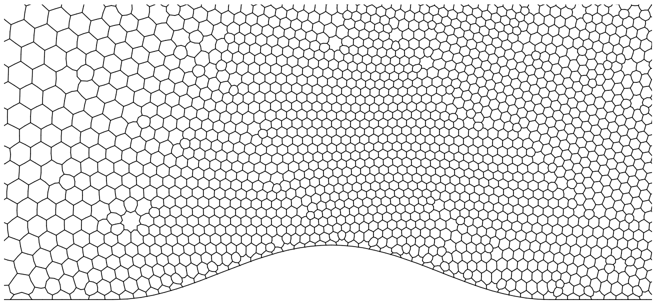
The Edge Based Finite Volume Discretisation







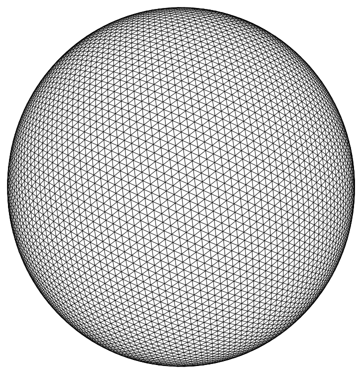
Edges

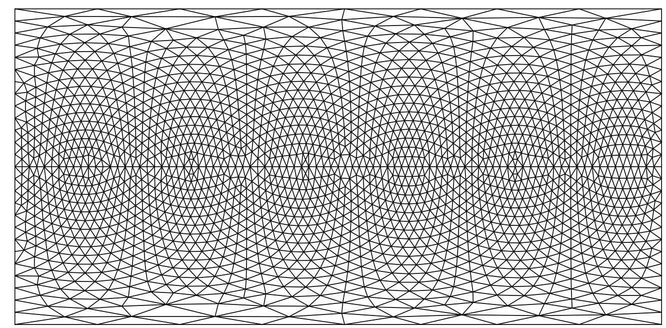


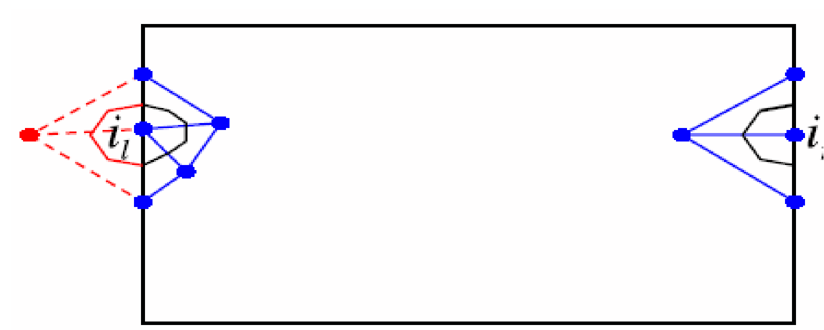
Median dual computational mesh
Finite volumes

Geospherical framework

$$\frac{\partial G\Phi}{\partial t} + \nabla \cdot (\mathbf{V}\Phi) = GR \qquad \Phi_i^{n+1} = \mathcal{A}_i(\Phi^n + 0.5\delta t R^n,\ \mathbf{V}^{n+1/2}, G) + 0.5\delta t R_i^{n+1}$$

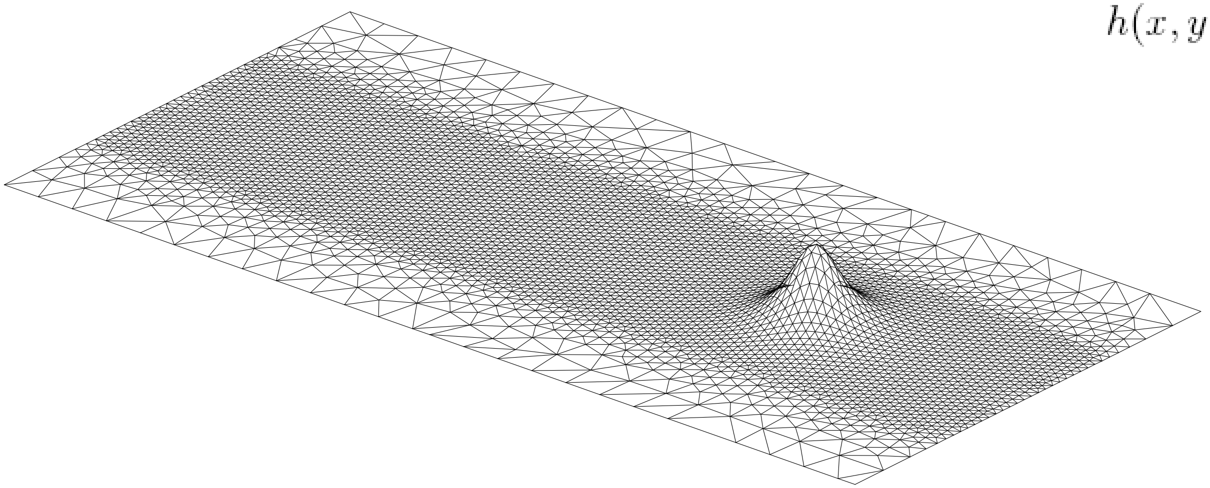






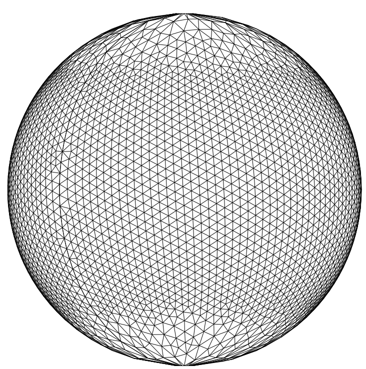
(Szmelter & Smolarkiewicz, J. Comput. Phys. 2010)

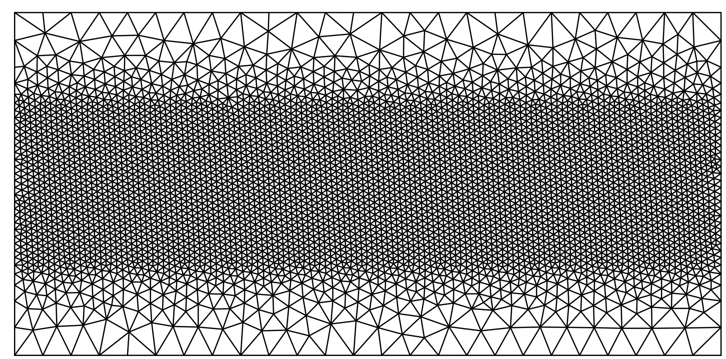
A stratified 3D mesoscale flow past an isolated hill



$$h(x, y) = h_0[1. + (l/\mathcal{L})^2]^{-3/2},$$

(4532 points)



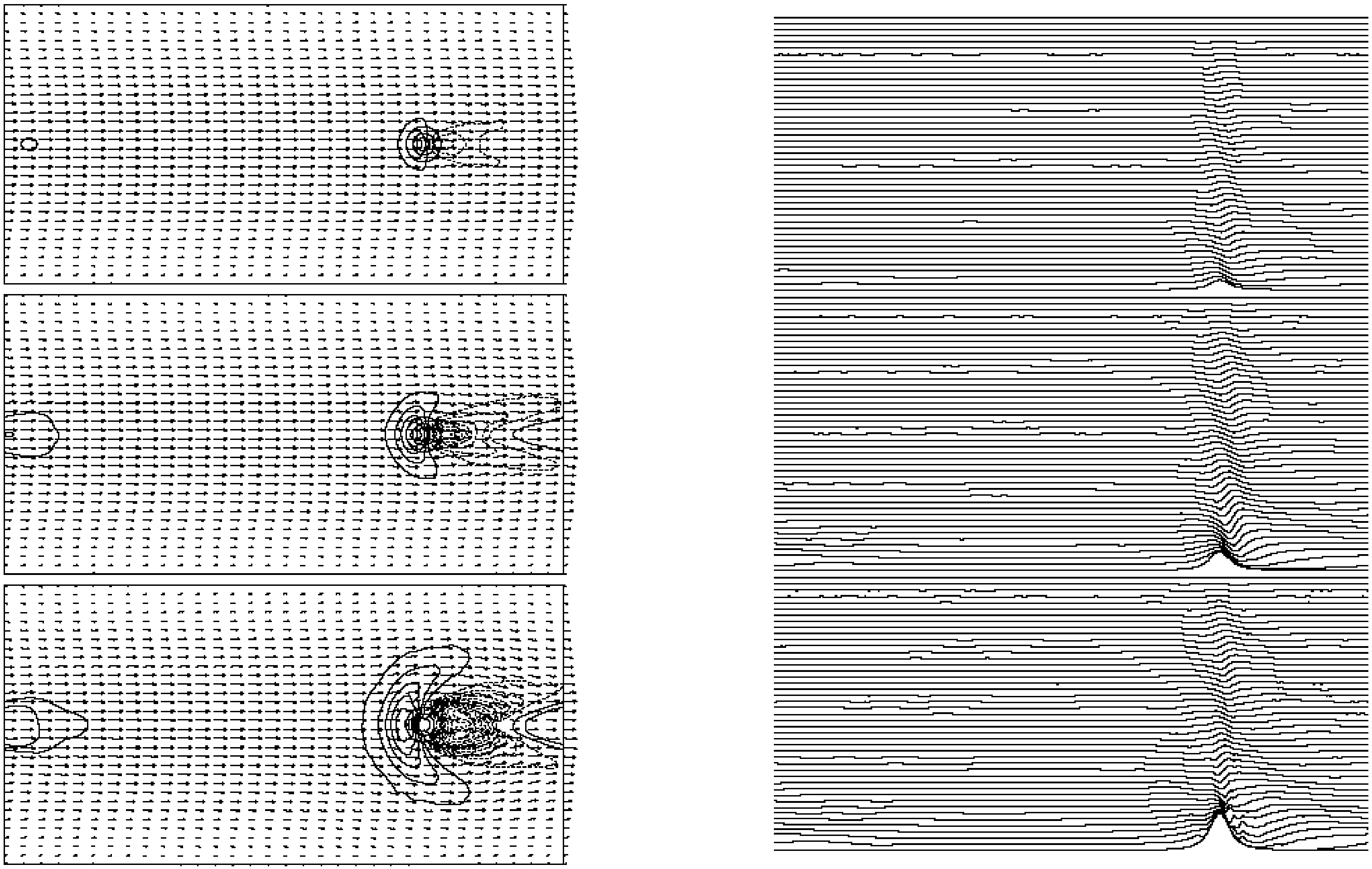


Reduced planets (Wedi & Smolarkiewicz, QJR 2009)

Stratified (mesoscale) flow past an isolated hill on a reduced planet

4 hours

$Fr = U_0/Nh$



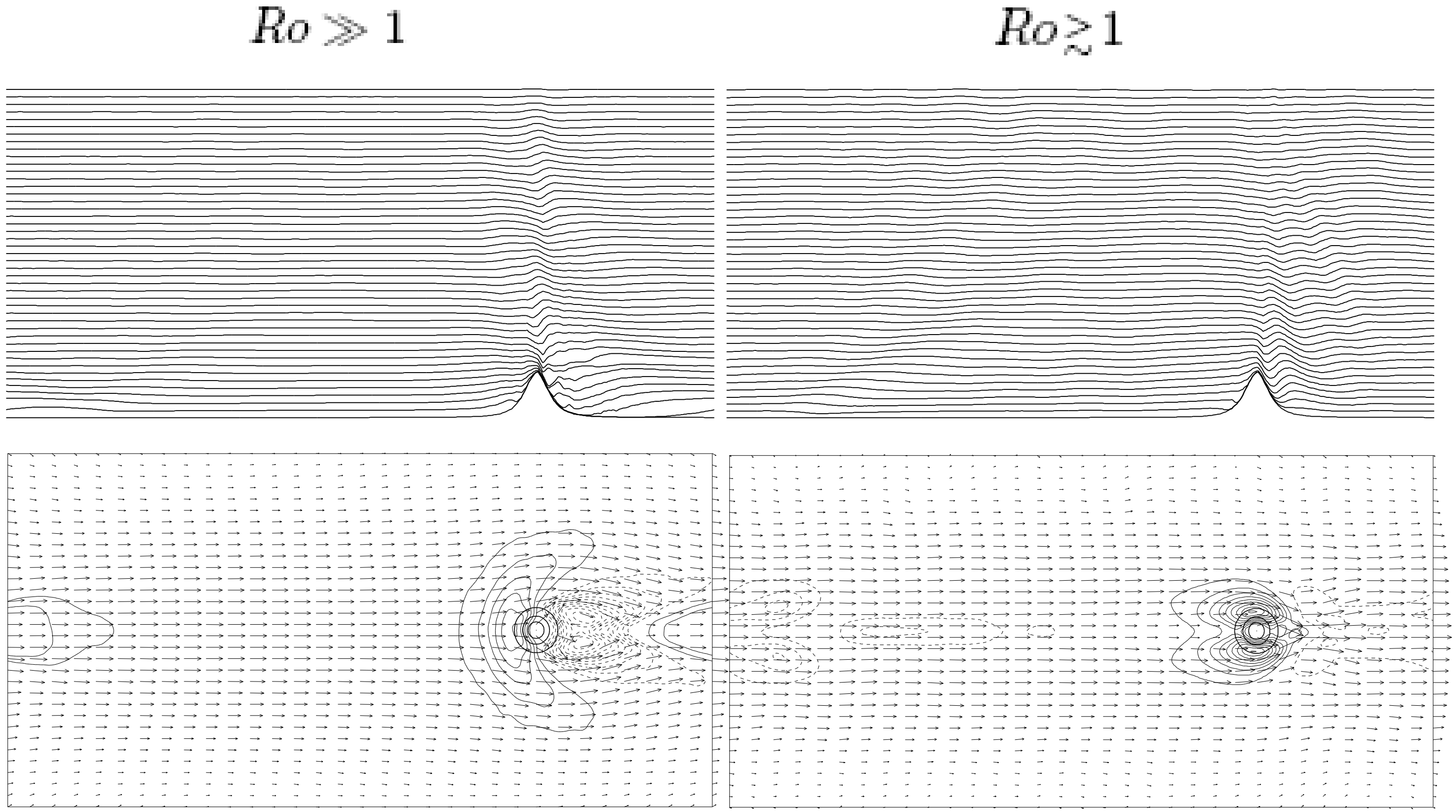
Fr=2

Fr=1

Fr=0.5

Hunt & Snyder J. Fluid Mech. 1980; Smolar. & Rotunno, J. Atmos. Sci. 1989 ; Wedi & Smolar., QJR 2009

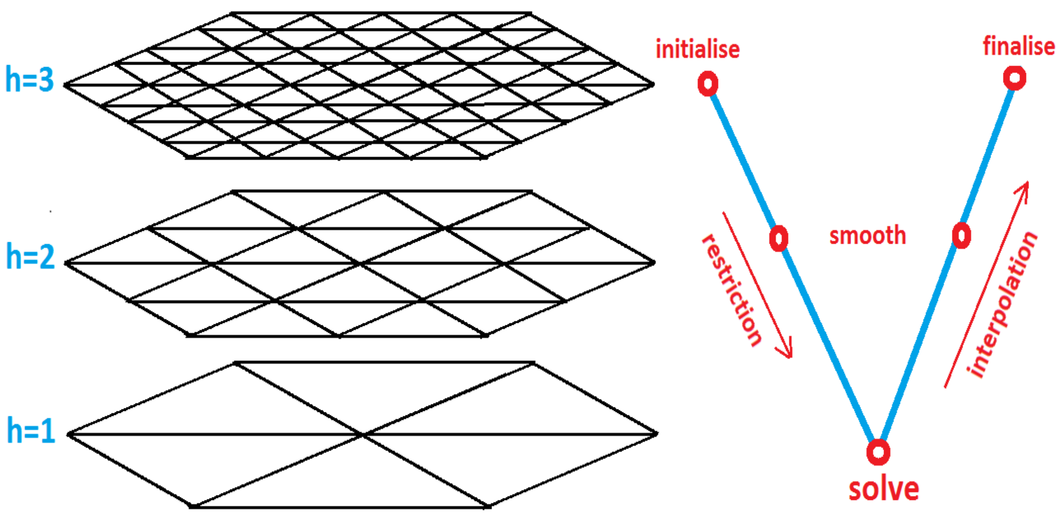
Fr=0.5

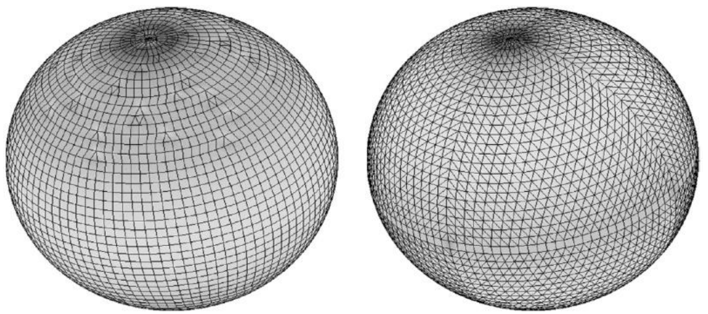


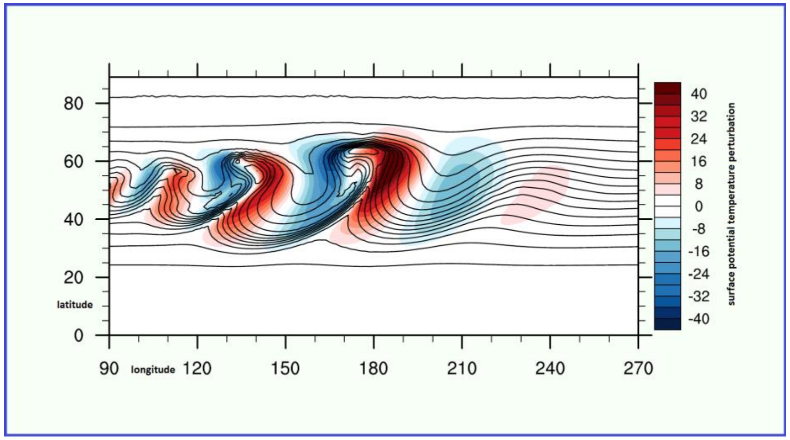
Smith, Advances in Geophys 1979; Hunt, Olafsson & Bougeault, QJR 2001

Multigrid techniques

V-Cycle



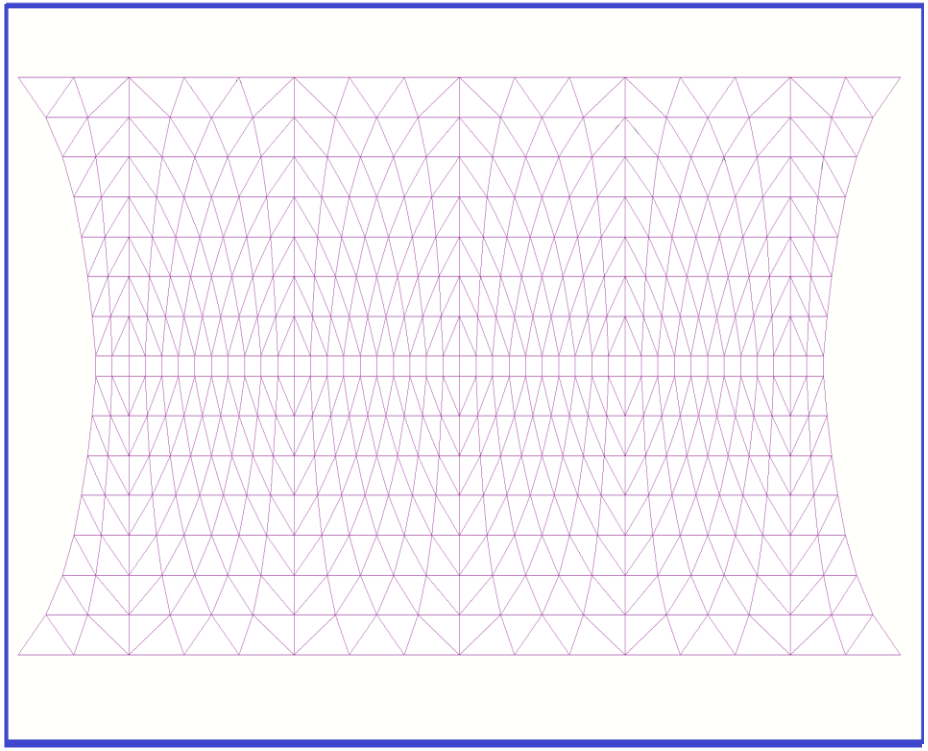




Multigrid using Atlas

Octahedral 16 mesh: Remove odd latitudes

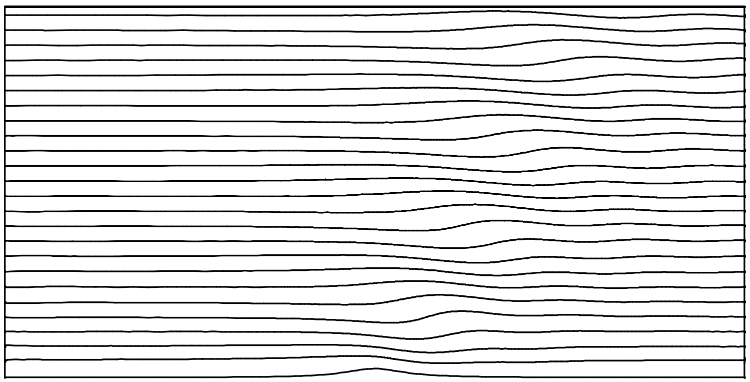
computational domain Single level of mesh coarsening



Nonhydrostatic Boussinesq mountain wave

Szmelter & Smolarkiewicz , *Comp. Fluids*, 2011

$$\nabla \bullet (\mathbf{V}\rho_o) = 0 \ ,$$
$$\frac{\partial \rho_o V^I}{\partial t} + \nabla \bullet (\mathbf{V}\rho_o V^I) = -\rho_o \frac{\partial \tilde{p}}{\partial x^I} + g\rho_o \frac{\theta'}{\theta_o}\delta_{I2}$$
$$\frac{\partial \rho_o \theta}{\partial t} + \nabla \bullet (\mathbf{V}\rho_o \theta) = 0 \ .$$



$NL/U_o = 2.4$

Comparison with the EULAG's (structured mesh) results --- very close
with the linear theories (Smith 1979, Durran 2003):
over 7 wavelenghts : 3% in wavelength; 8% in propagation angle; wave amplitude loss 7%

$$\frac{\partial \Phi}{\partial t} + \nabla \bullet (\mathbf{V}\Phi) = \mathbf{R}$$

Gravity wave breaking in an isothermal stratosphere

$$\nabla \cdot (\bar{\rho}\mathbf{v}) = 0\,, \quad \frac{D\theta}{Dt} = 0\,, \quad \frac{D\mathbf{v}}{Dt} = -\nabla\Phi' - \mathrm{g}\frac{\theta'}{\bar{\theta}}\,,$$ *Lipps & Hemler*

$$\nabla \cdot (\bar{\rho}\bar{\theta}\mathbf{v}) = 0\,, \quad \frac{D\theta}{Dt} = 0\,, \quad \frac{D\mathbf{v}}{Dt} = -c_p\theta\nabla\pi' - \mathrm{g}\frac{\theta'}{\bar{\theta}}$$ *Durran*

$$D\psi/Dt = R$$

by combining $\rho^* \cdot \left(D\psi/Dt = R\right)$ with $\psi \cdot \left(\nabla\rho^*\mathbf{v} = 0\right)$,

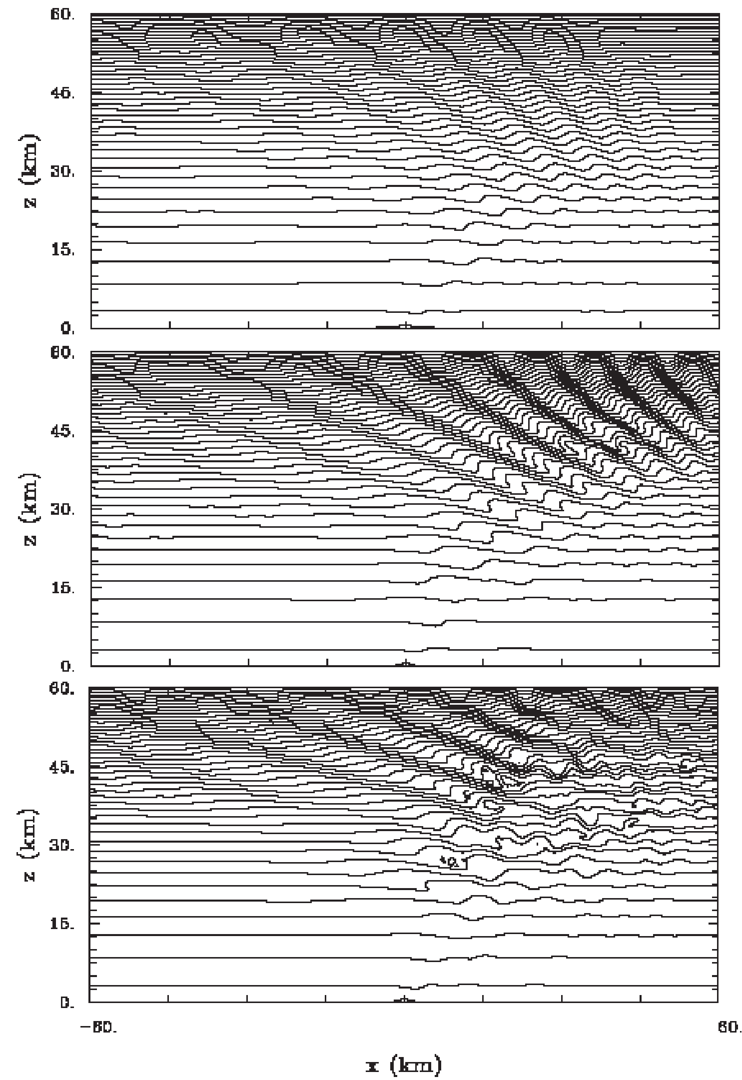
$$\frac{\partial \rho^* \psi}{\partial t} + \nabla \cdot (\rho^* \mathbf{v}\psi) = \rho^* R\,.$$

$$\psi_i^{n+1} = \mathcal{A}_i(\tilde{\psi}, \mathbf{v}^{n+1/2}, \rho^*) + 0.5\delta t R_i^{n+1}$$

$$S_\theta \;=\; d\ln\bar{\theta}/dz \;=\; 4.4\cdot 10^{-5}\ \mathrm{m}^{-1}$$

$$\mathbf{v}_e \;=\; (u_e, 0) \quad u_e \;=\; U \;=\; 20\mathrm{ms}^{-1}$$

Isentropes at $t = 60$, 90, and 120 min.

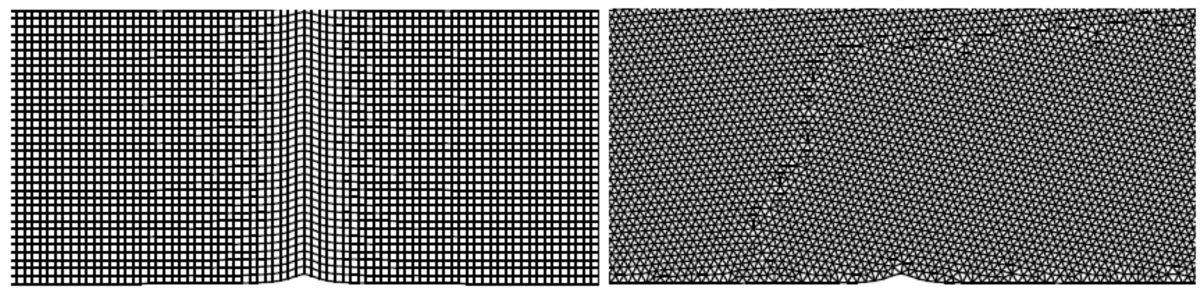


(Prusa et al JAS 1996,
Smolarkiewicz & Margolin, Atmos. Ocean 1997
Klein, Ann. Rev. Fluid Dyn., 2010,
Smolarkiewicz et al Acta Geoph 2011)

Smolarkiewicz et al
Acta Geoph 2011

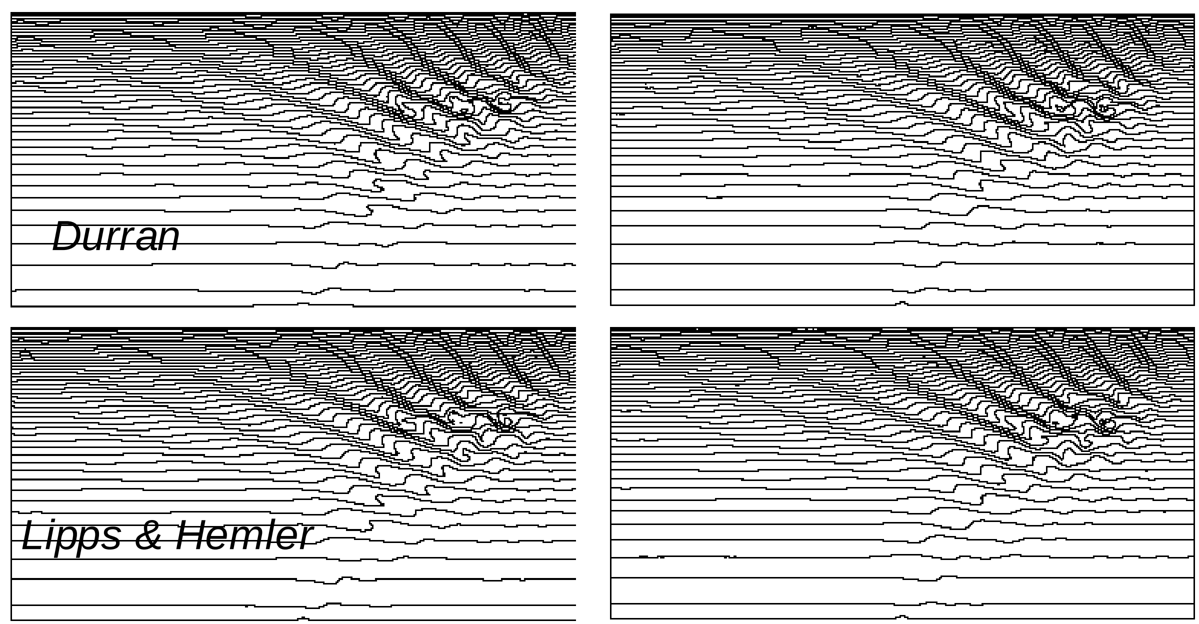
Gravity wave breaking in an isothermal stratosphere

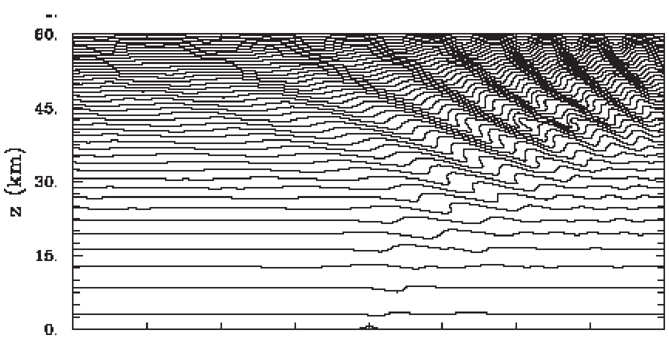
Nonhydristatic Edge-Based NFT



Isentropes at $t = 90$ min

EULAG CV/GRID

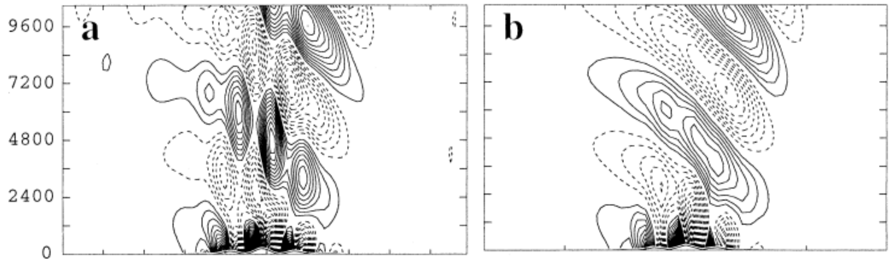


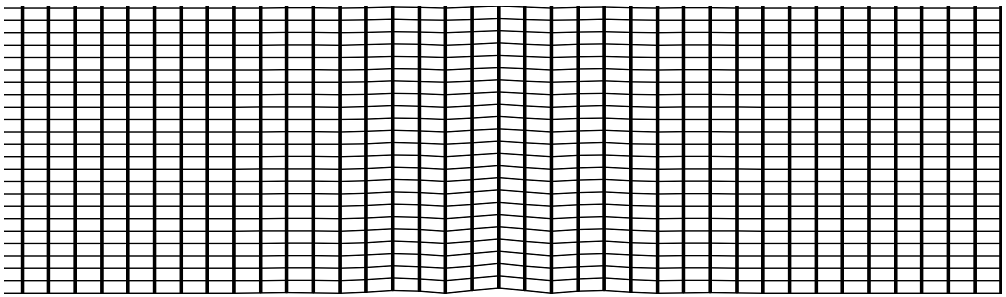


Static mesh adaptivity with MPDATA based error indicator

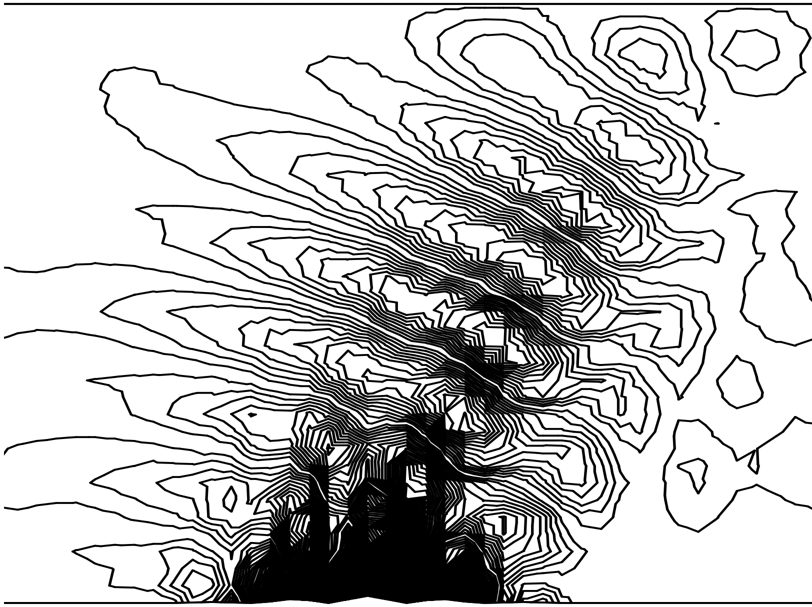
Schär Mon. Wea. Rev. 2002

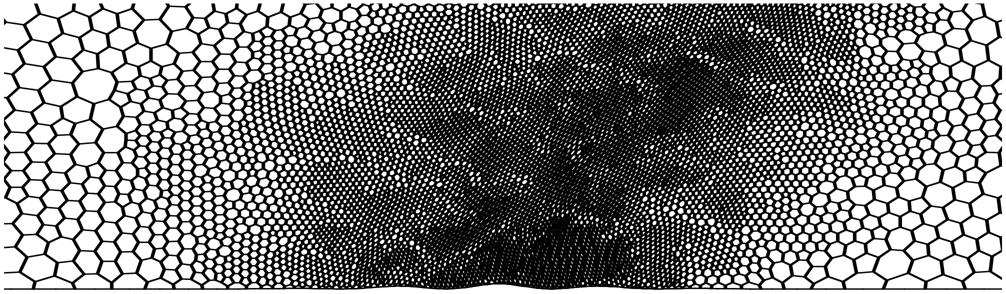
(Recommended mesh ca10000 points)





Coarse initial mesh 80x45 =3600 points and solution



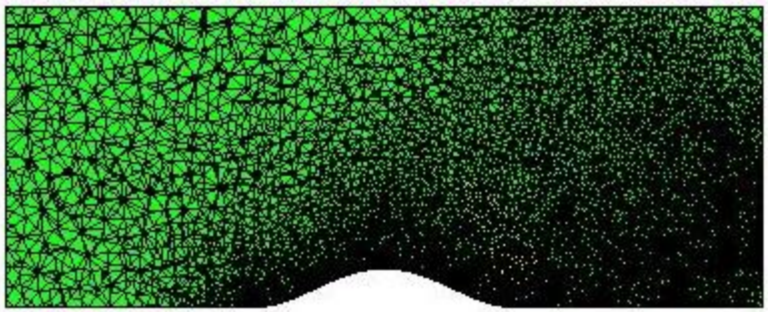


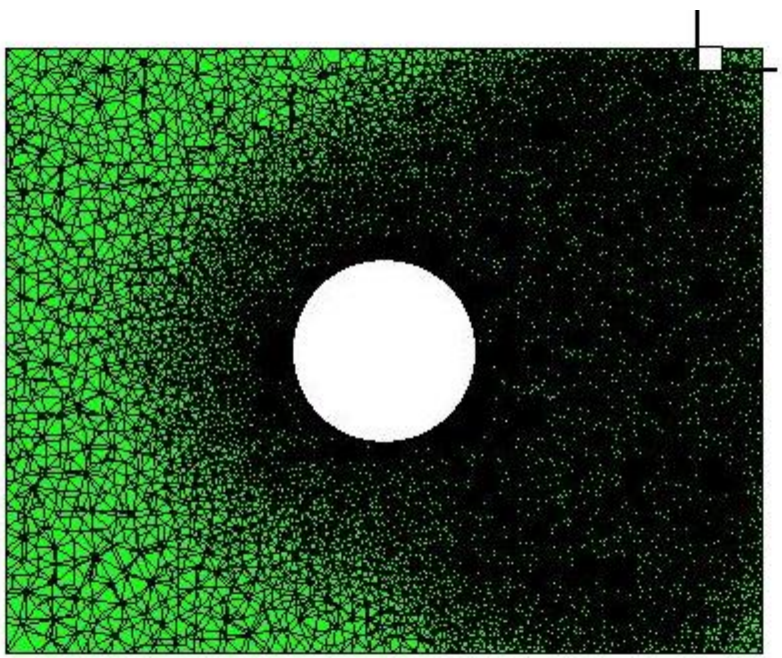
Adapted mesh 8662 points and solution

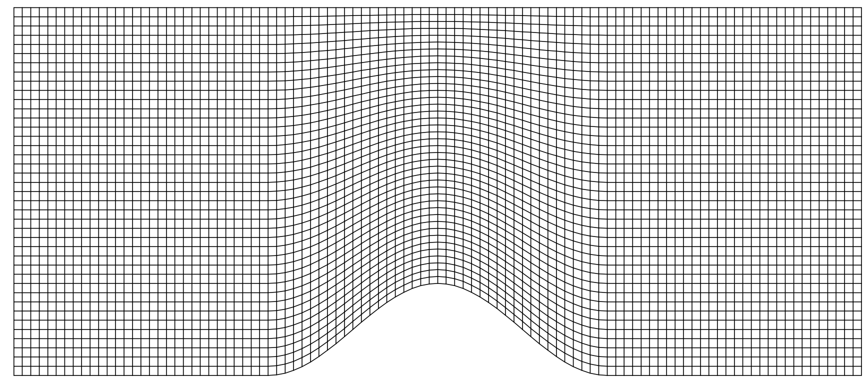
Szmelter et al JCP 2015

Stratified flow past a steep isolated hill

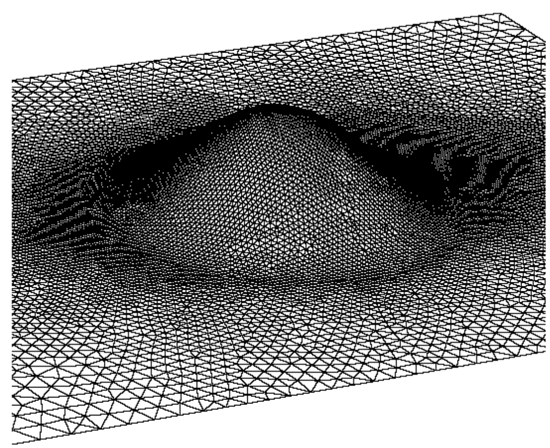
1121192 Cartesian dx=100
692533 Distorted prisms dx=100-400
441645 tetra dx=50 -450

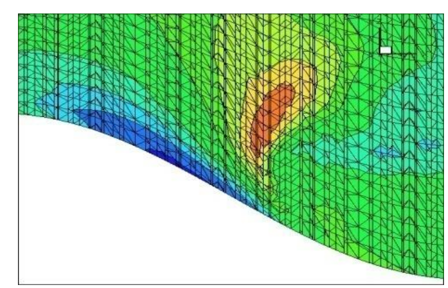






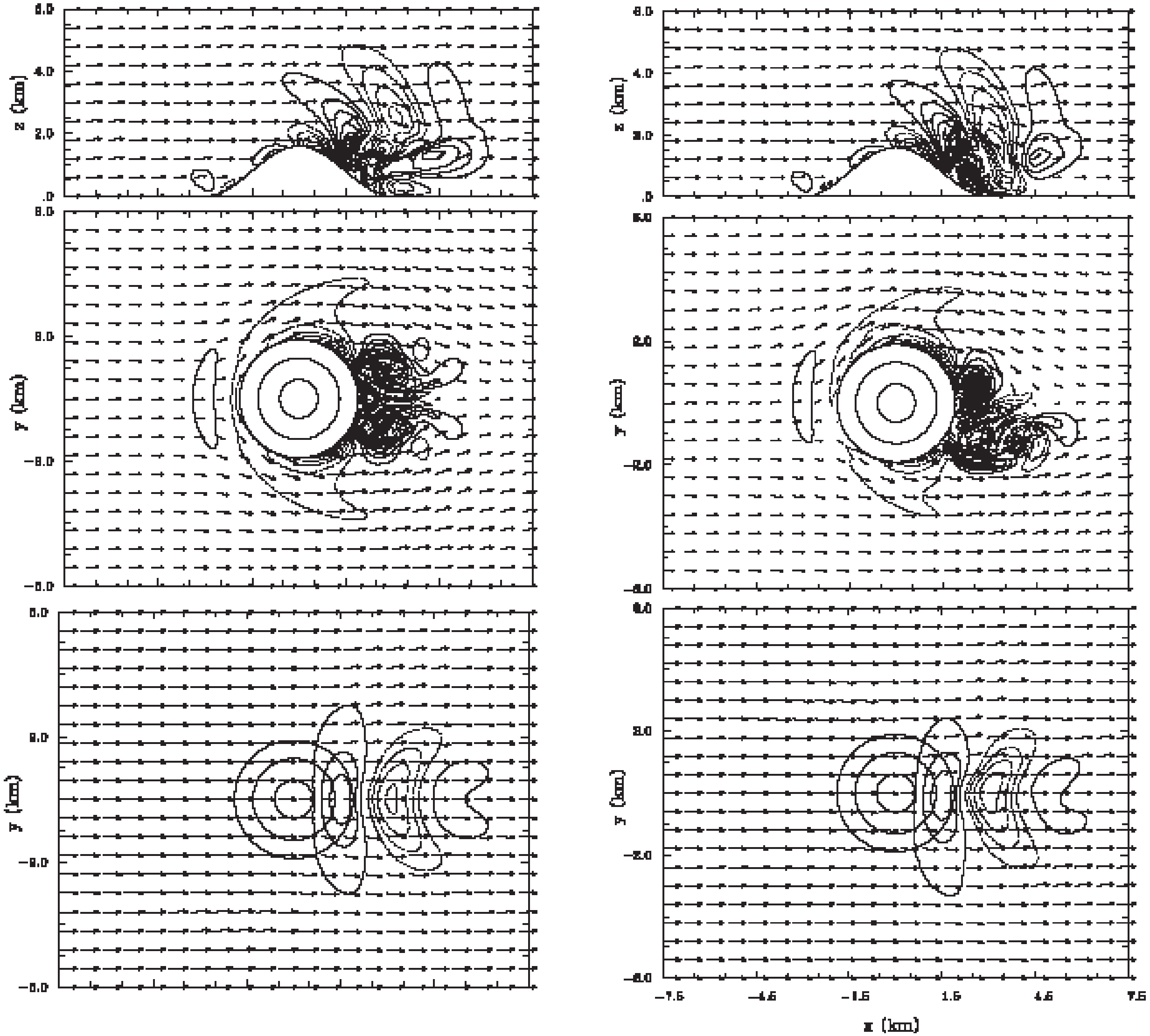
$$z_{i,k} = (k-1)\delta z \left(1 - \frac{h_i}{H}\right) + h_i \ ,$$





Stratified flow past a steep isolated hill

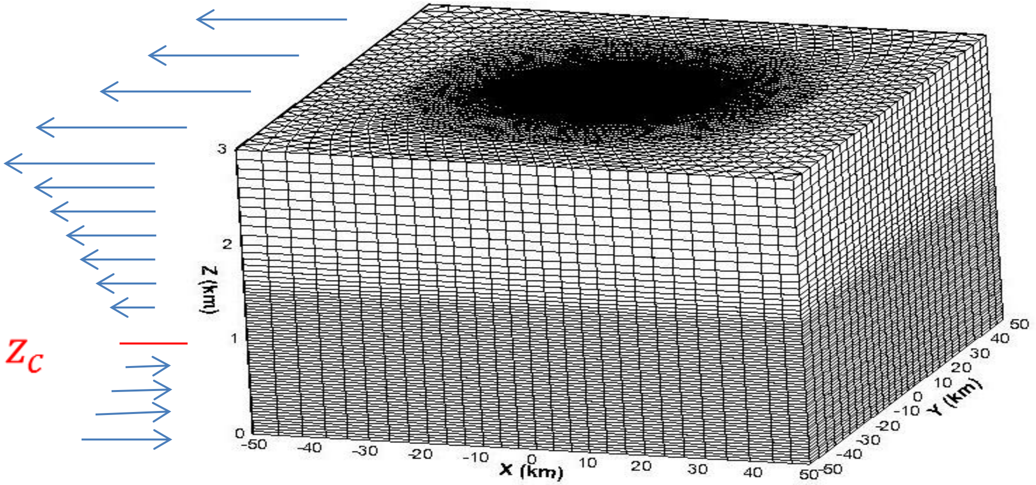
Hunt & Snyder JFM 1980
Smolarkiewicz & Rotuno JAS 1989



$Fr = 1/3, \ Ro \nearrow \infty$ $Fr = 1/3, \ Ro \approx 3$

The effect of critical levels on stratified flows past an axisymmetric mountain

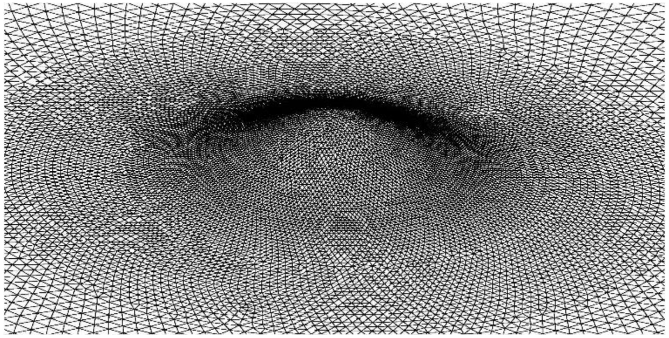
Szmelter et. al. JCP 2015



Prismatic mesh

$$\tilde{z}_{i,k} = \tilde{z}_{i,k-1} + \delta \tilde{z}_k$$

$$\tilde{z}_{i,k} = \tilde{z}_{i,k}\left(1 - \frac{h_i}{H}\right) + h_i$$



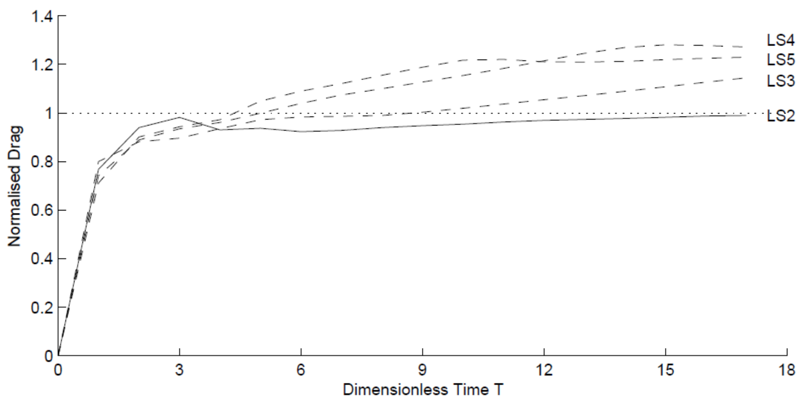
Triangular surface mesh

$$U_0 = 10\, m s^{-1}\,, \qquad U(z) = U_0\left(1 - \frac{z}{z_c}\right)$$

$$\frac{U_0}{N_a} = 0.2\,, \qquad R_i = \left(\frac{N z_c}{U_0}\right)^2 = 1\,, \qquad N = 0.01\, s^{-1}$$

$$a = 5000\, m\,, \qquad h(r) = h_0\left(1 + \frac{r^2}{a^2}\right)^{-\frac{3}{2}}\,, \qquad r \equiv \sqrt{x^2 + y^2}$$

The effect of critical levels on stratified flows past an axisymmetric mountain



$$\hat{h} = \frac{h_0 N}{U_0} \qquad T = \frac{tU_0}{a}$$

Experiment	Ri	$\hat{h}$	$D/D_0(6)$	$D/D_0(10)$	$D/D_0(18)$
LS2 (linear)	1	0.05	0.923	0.954	0.997
LS3 (nonlinear)	1	0.1	0.983	1.02	1.160
LS4 (nonlinear)	1	0.2	1.04	1.15	1.26
LS5 (nonlinear)	1	0.3	1.09	1.22	1.23

Grubisic and Smolarkiewicz J. Atmos Sci 1997

The effect of critical levels on stratified flows past an axisymmetric mountain

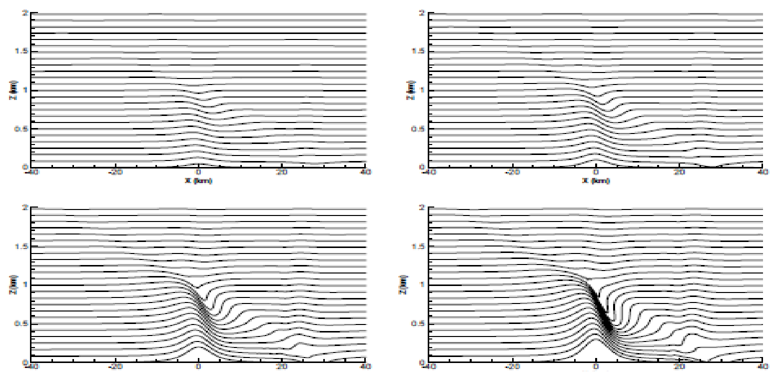


Fig. 12. Isentropes at T=6 in $y = 0$ vertical plane for experiments LS2 (left, top) LS3 (right, top), LS4 (left, bottom) and LS5 (right, bottom).

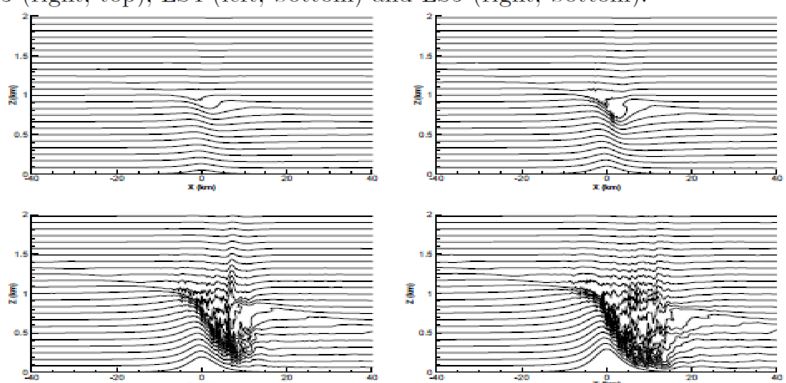
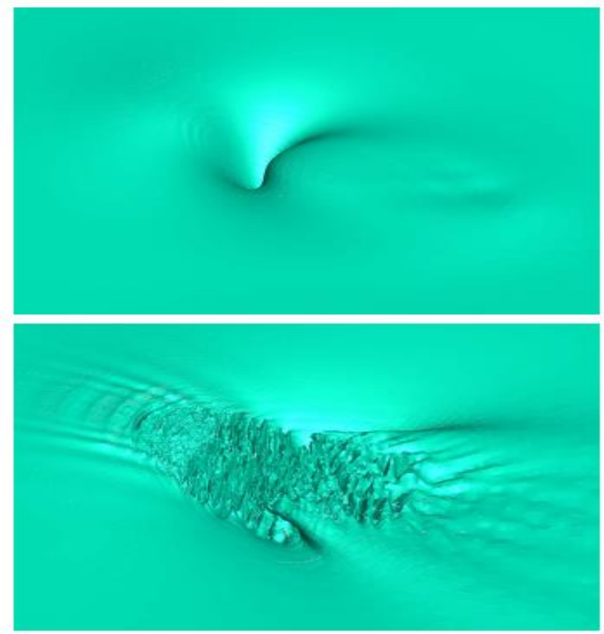


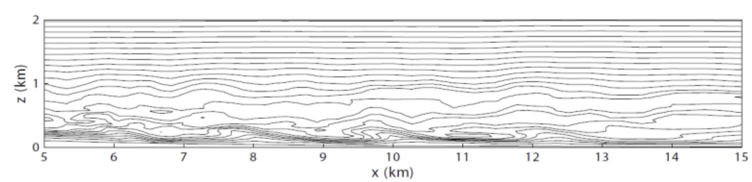
Fig. 13. As in Fig. 12 but at T=18.



$T = 6$

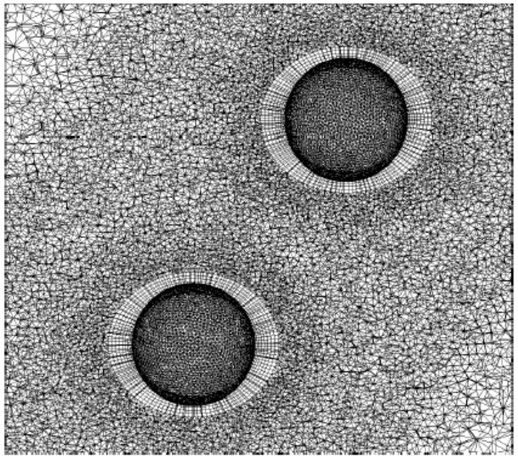
$T = 18$

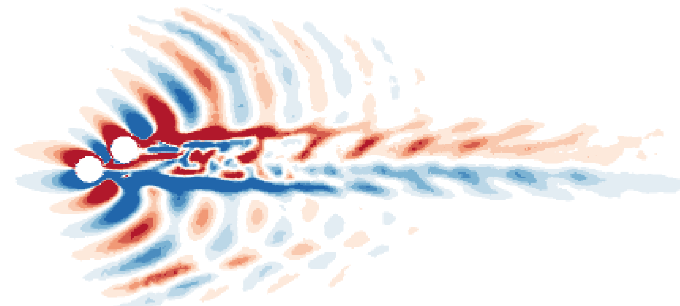
the isentrope with undisturbed height $z = 0.94z_c$ LS5.



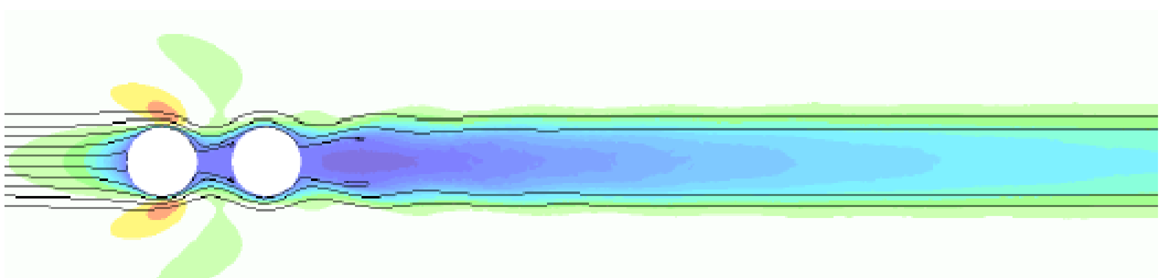
Szmelter et al JCP 2015

Stratified flow past two spheres,
Fr=0.625, Re=300

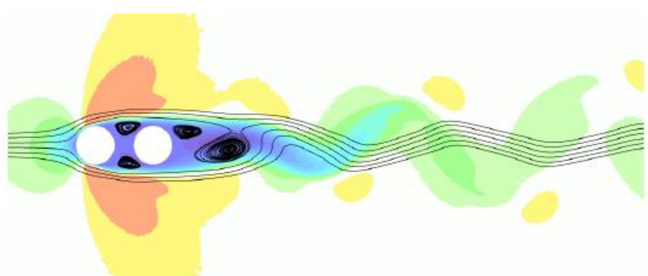




Fr=0.625
Tilted configuration



Fr=0.625



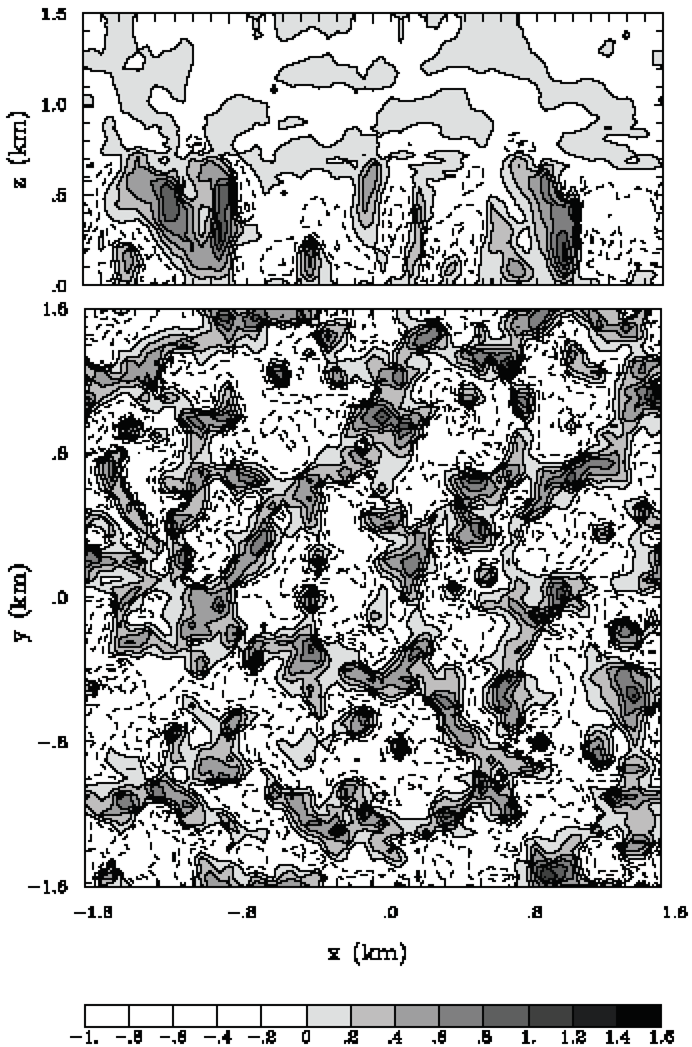
Fr=0.25

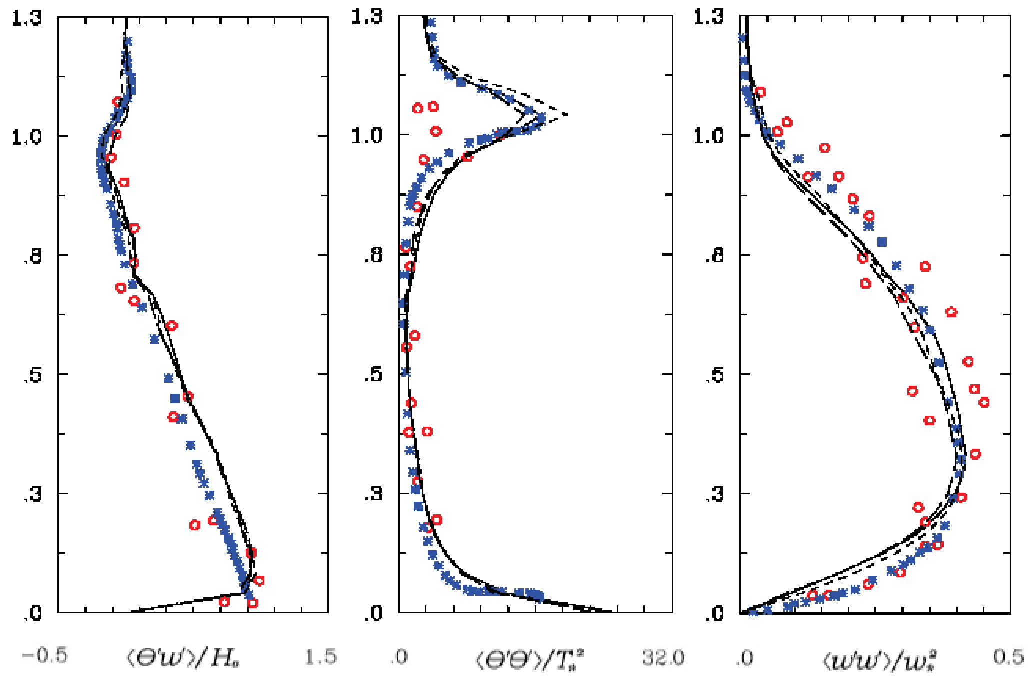
Cocetta et al PoF 2021

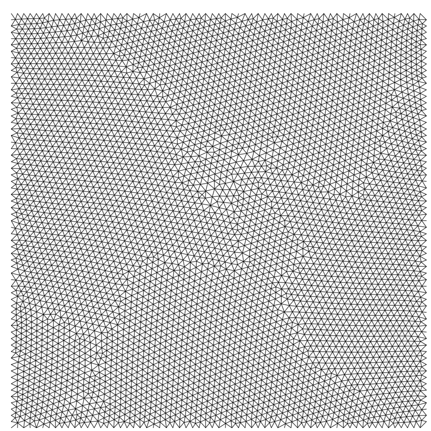
Convective Planetary Boundary Layer

Schmidt & Schumann JFM 1989

Smolarkiewicz et al JCP 2013







Edge-based T

Edge-based C

EULAG

SS LES

Observation

64x64x51

Selected references for further details

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