CARLOS GEIJO GUERRERO, AEMET (Spanish State Meteorological Agency)
cgeijog@aemet.es

ECMWF 50th Anniversary Seminar, Bonn, 9th April 2025

Variational Constraints for Convective Data Assimilation *and* An Interesting Correspondence between Covariances and Green Functions

Noise in the initial conditions becomes an important issue for nowcasting NWP applications. Within the HIRLAM Project a technique based on variational methods to filter this noise was developed (1) (2). It uses the model semi-implicit dynamics to give a precise definition to some constraints imposed on the analysis increments. In brief, let M represent this semi-implicit dynamics, Δx^k the analysis increments, and d^k the "fg – ob" differences, with superscript "k" indicating horizontal wavenumber. The filter is obtained by minimization of the cost : (w_o^k, w_c^k are free parameters that give the relative weight among the forcing and constraint terms)

$$2J(\Delta x^k) = \int_{bottom}^{top} w_o^k\,(\Delta x^k - d^k)^2 + w_c^k\,(M\Delta x^k)^2$$

Euler-Lagrange's equations for this problem are (k superscript omitted)

$$M^+M\,\Delta x + w\,\Delta x = w\,d; \quad w = \frac{w_o}{w_c};$$

$M^\dagger M$ is actually a 4 x 4 matrix operator, but it can be reduced to "Lower Triangular" form. The solution proceeds by back-substitution from the vertical velocity filtered increment, which satisfies the "Boundary Value Problem" on the ξ vertical coordinate

$$O_\xi(\Delta w) = \left[\partial_\xi^4 - a\,\partial_\xi^2 + b\right](\Delta w) = f\left(d_w, d_D, d_T, d_{\pi_s}\right);\ a,b \in \mathbb{R} > 0$$
$$\Delta w(top) = \Delta w(bottom) = \partial_\xi \Delta w(top) = \partial_\xi \Delta w(bottom) = 0$$

GREEN FUNCTIONS AND COVARIANCES

The VC algorithm and the 3DVar algorithm are both formulated as the stationary point of a quadratic form. The latter has solution $B^{-1}\,\Delta x + R^{-1}(\Delta x - d) = 0$ which bears an evident similitude with the Euler-Lagrange equations for the VC solution. This immediately suggests the correspondence

$$M^\dagger M \quad \Leftrightarrow \quad B^{-1} \quad hence? \quad GF_{O_\xi} \Leftrightarrow \quad B$$

To put this correspondence on more firm footing, it is necessary to check weather the VC GF possesses the same properties that characterize auto-covariances, namely, symmetry and definite (positive) norm. The symmetry of the GF , that is, $GF_{O_\xi}(\xi,\eta) = GF_{O_\xi}(\eta,\xi)$, follows from the self-adjoint character of O_ξ . The positive definite condition is also readily confirmed by the fact that O_ξ factorizes as follows

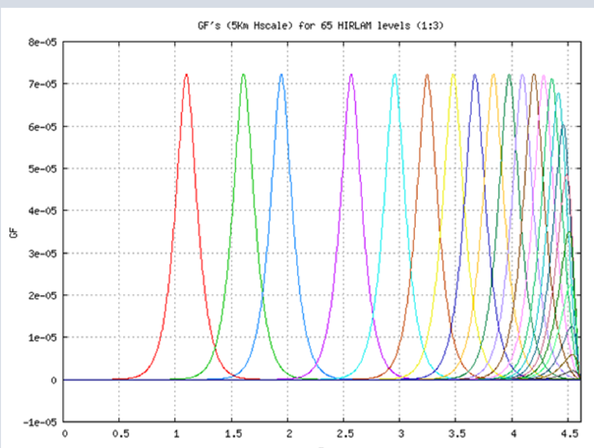
$$O_\xi = K^\dagger K \quad ; \quad K = \partial(\partial+1) - \lambda \equiv L - \lambda \quad ; \lambda \in \mathbb{R} > 0$$

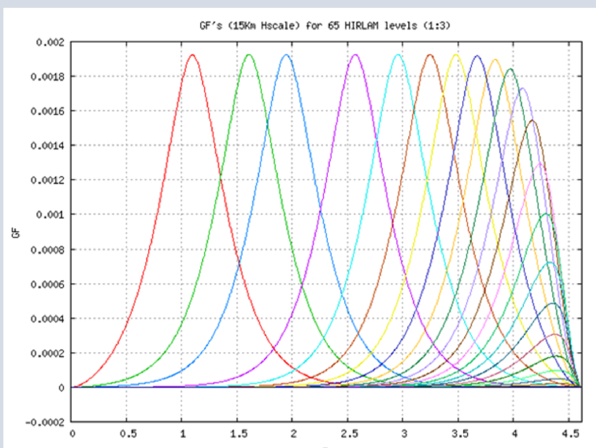
and then $\langle u, Ou\rangle = \langle u, K^\dagger K u\rangle = \langle Ku, Ku\rangle > 0\ \forall u$ when $K\,u = 0 \quad iff \quad u = 0$, that is , K has empty kernel. This is so in the VC algorithm because with the imposed boundary conditions u(top)=u(bottom)=∂u(top)=∂u(bottom)=0, and λ>0 and real , λ cannot be in the spectrum of L.

The Green's function for the VC algorithm is therefore a genuine auto-covariance operator. We now turn the argument round and ask weather we can improve on the construction of the 3DVar **B** by considering it as something that can function as a GF. At this point, it may come to one's mind that in some areas of Theoretical Physics, GFs for "free random (quantum) fields" ("propagators" in that jargon) can be modulated by external or ambient fields by introducing suitable modifications in the variational formulation of the dynamics of these fields. Just below this line I show an application of this idea.

GREEN FUNCTIONS FOR THE VARIATIONAL CONSTRAINTS (VC) ALGORITHM

O_ξ is a self-adjoint fourth order differential operator in the vertical downwards running coordinate ξ and its Green Function (GF) can be easily calculated (2). These GF kernels are a key ingredient of the numerical technique used to solve by quadrature the variational constraints problem. The figure below displays two families of these GF kernels for different k wavenumbers (5 and 15 Km left and right respectively). One of the conspicuous features of these curves is the direct relationship between vertical and horizontal scales, exactly as it happens with the 3DVar "structure functions" for the constraint **B**. We also notice that they are strictly positive.

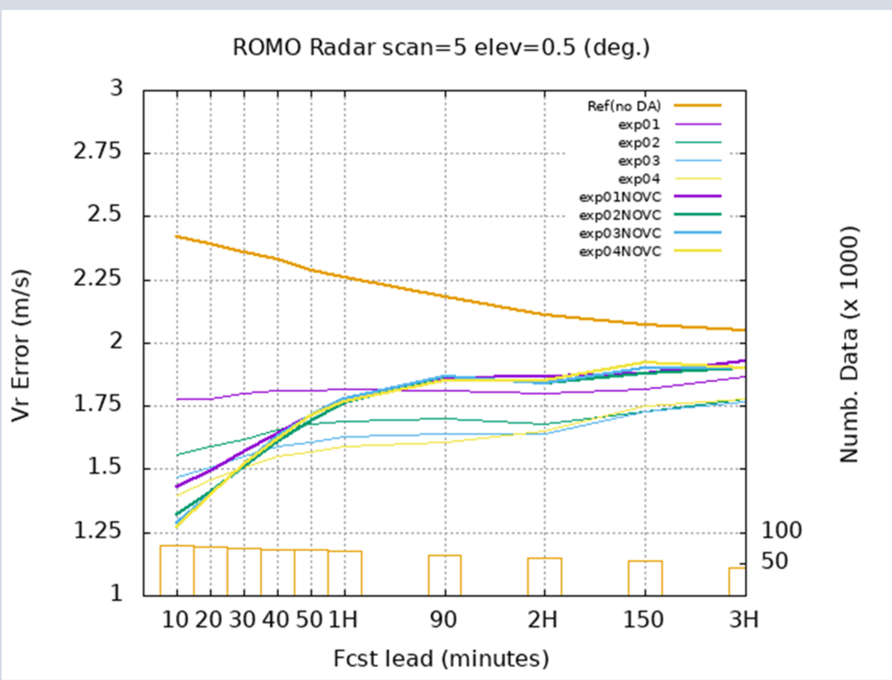




INITIALIZATION EXPERIMENTS WITH SUB-HOURLY DA OF RADAR WIND DATA

10-minutes volumes of wind data from the Danish Römö weather radar collected during the early hours of 23rd May 2023 were used in the initialization of HARMONIE-AROME LAM NWP system with 10-minutes DA cycles (3). An important set-up parameter in these experiments was the number of consecutive 10-minutes cycles or "model wind-up time interval" employed. Configurations with up to four consecutive cycles, that is, wind-up intervals up to 30 minutes were tested. The 3D-Var multivariate increments were submitted to the VC filter in order to supress noise in the initial conditions, and is precisely the impact of this last processing step the main aspect to be analysed in these experiments.

On the right the error growth curves are shown. In the NO-VC experiments the error increases fast and levels off after about one hour, while in the VC experiments the departure from observations augments at a clearly slower rate. In the NO-VC experiments the initial state draws closer towards the observations but the subsequent evolution of the forecast indicates that the fit contained much noise (overfitting). Also, the curves for the different experiments (0,10,20 and 30 minutes wind-up intervals) are almost indistinguishable in the NO-VC case, while they are well separated and monotonous decreasing for the VC case, which means that successive assimilations acted coherently, not so in the NO-VC case.



(1) C. Geijo , P. Escribá "Variational Constraints for DA in ALADIN-NH Dynamics" .ALADIN-HIRLAM NewsLetter#11, August 2018 https://www.umr-cnrm.fr/aladin/IMG/pdf/nl11.pdf
(2) C. Geijo , "Annex to Variational Constraints for DA in ALADIN-NH Dynamics". August 2018 https://www.researchgate.net/publication/327117950_ANNEX_to_Variational_Constraints_for_DA_in_ALADIN-NH_Dynamics
(3) C. Geijo , "Exploring Sub-hourly DA in HARMONIE-AROME". ACCORD NewsLetter#4, June 2023 https://www.umr-cnrm.fr/accord/IMG/pdf/accord-nl4.pdf
(4) C. Geijo , "Scientific Note for ACCORD RWP2025". In preparation. To be published soon in ACCORD NewsLetters

MODELLING FLOW-DEPENDENT COVARIANCES WITH GAUSSIAN INTEGRALS

Let Δ be an N-dimensional random variable. We can put this N-dimensional variable on an N-node grid and, following the analogy, think the covariance $\langle \Delta_i \Delta_j \rangle$ as the GF that links the values that Δ takes on the two grid nodes i and j. Let's say that $\langle \Delta_i \Delta_j \rangle$ "propagates the signal" from i to j. It is as if the Δ field has got now some (stationary) dynamics. Methods in QFT (known as radiative corrections) show how the "propagation from one place to another" has to be modified when a background force breaks homogeneity of the space-time in which that propagation takes place: suitable interactions between Δ and the background have to be added to the kinetic term in the lagrangian.

Let then Δ be the random model error field on the N-grid. Baseline assumptions in operational NWP take frequently this random error with Gaussian distribution and homogeneous over the grid. The B^{-1} term in the exponential is now the kinetic term (below in blue), represents the "free propagation" without external forces. For the interaction with an ambient field $\vec{V}$ that introduces the flow-dependency there are many possibilities at modeller's disposal. Some choices are however more convenient than others at the time of carrying out actual computations. There are also more delicate considerations on guarantee of existence of the calculations to be done. One very convenient choice is the sum over the N-nodes of squares of error advection by $\vec{V}$ (in red). The sought covariances can then be computed from this expression

$$\langle \Delta_i \Delta_j \rangle \left[\vec{V}\right] \ \propto \ \int \ldots \int d\Delta_1 \ldots d\Delta_N\ \Delta_i \Delta_j \exp\left(-\frac{1}{2}\Delta^T B^{-1} \Delta - \frac{\mu}{2} tr\left(\left[\vec{V}\bullet\vec{\nabla}\Delta\right]\left[\vec{V}\bullet\vec{\nabla}\Delta\right]^T\right)\right)$$

IMPLEMENTATION OF A PROTOTYPE AND TEST

It turns out to be advantageous to carry out the computations in wavenumber space (4). The algorithm can then be expressed as a power series in the correction term (or in μ)

$$\langle \Delta_k \Delta_l \rangle = B_k \delta_{k,l} - \mu\, M_{k,-l}\, B_k B_l + \mu^2 \left(\sum_m B_m\, M_{k,m}\, M_{m,-l}\right) B_k B_l - \mu^3 \left(\sum_{m,n} B_m\, B_n\, M_{k,m}\, M_{m,n}\, M_{n,-l}\right) B_k B_l + \ldots$$

where $M_{k,m}$ is the k-representation of the correction term, computed from the spectral components of $\vec{V}$

$$M_{k,l} \equiv M\left(\vec{k},\vec{l}\right) = \frac{1}{N}\sum_{\vec{G}} \left(\vec{s}_{\vec{k}}\, \vec{V}^*_{\vec{G}-\vec{k}}\right)\left(\vec{s}_{\vec{l}}\, \vec{V}_{\vec{G}-\vec{l}}\right) \ ; \ \vec{k} = \frac{2\pi}{\sqrt{N}}(k_x, k_y) \ ; \ \vec{s}_{\vec{k}} = \left(\sin(\frac{2\pi}{\sqrt{N}}k_x),\ \sin(\frac{2\pi}{\sqrt{N}}k_y)\right) \ (k_x, k_y) \in \mathbb{Z}\{0,1\ldots\sqrt{N}-1\};$$

The figure below shows the results of a test on a domain 128 x 128, with a Δ-correlation length of about 6 points and wind field with complex structure. The method modulates the covariances as expected according to the flow. This exercise is then a successful "proof of concept" of the idea. An important aspect still to be work out in more detail is the feasibility of its use in operational environments with strong performance constraints.